

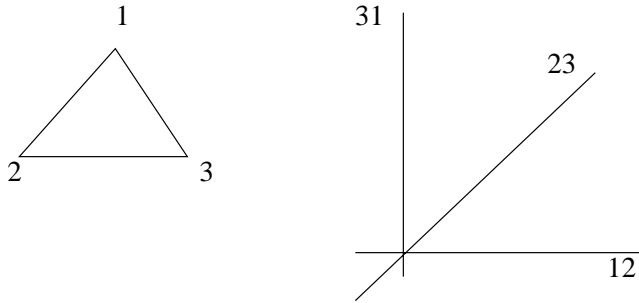
18.433 Combinatorial Optimization

Polyhedral Combinatorics

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Lecturer: Santosh Vempala

So far we have treated graphs as sets of vertices and edges, $G=(V,E)$. One can also think of each edge as an axis. Any point in space corresponds to a graph. The coordinates determine the edge weight.



The point $(1,0,1)$ in this space would correspond to a graph with only 2 edges present (i.e. of weight > 0) : $(1,2)$ and $(3,1)$. How would we define a matching? It will also be a point with all coordinates being either 0 or 1. If $M = e_1, \dots, e_k$, then $X_m = (1, 0, 1, \dots, 0, 1)$, where the first coordinate indicates that $e_1 \in M$, the second coordinate indicates that $e_2 \notin M$, and so forth.

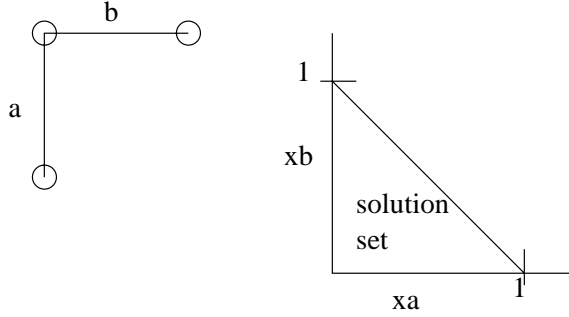
We think of matchings as solutions to equations (if it is a point). Consider $\mathbf{x} = (x_{e_1}, x_{e_2}, \dots, x_{e_m})$ as a vector of all edges in the graph.

1. $x_e \in \{0, 1\} \forall e \in E$
2. $\sum_{e \in \delta(v)} x_e \leq 1 \forall v \in V$

Claim 1. Any solution to these two equations is a matching.

Proof. If a vertex had more than one incident edge then it wouldn't be a matching. \square

Constraint (1) appears to be very strong. Suppose we replace it with the restriction that $0 \leq x_e \leq 1$. Now there are solutions that aren't matchings.



Consider the two edge graph shown above. The matchings here are $(0,1)$, $(1,0)$, and $(0,0)$. Note that the matchings bound the set of solutions (i.e. their convex hull equals the solution set). Is this always true? Are the “corners” always the matchings?

Consider the complete graph with 3 vertices, with all edge weights equal to one-half. This satisfies both equations. The corners of this shape will be at $(1,0,0)$, $(0,0,1)$, $(0,1,0)$, and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The last corner is not a matching, so it is not always true that the corners are the matchings.

Review of linear algebra and convexity

- $x_1, \dots, x_m \in \mathbb{R}^n$ (Each x_i is a vector with n coordinates)
- $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.
- A *linear combination* is $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m = \sum \lambda_i x_i$.
- An *affine combination* is a linear combination where $\sum \lambda_i = 1$.
- A *convex combination* is an affine combination where $\lambda_i \geq 0 \forall i$.

For example, given 2 points x_1 and x_2 , what points are convex combinations of them? The answer is the line segment between them. What points are affine combinations? The infinite line through the two points. What about linear combinations? The whole plane (in 3D, this would be the plane defined by x_1 , x_2 , and the origin).

The *linear hull* (span) of x_1, \dots, x_n is the set of vectors $\{\sum \lambda_i x_i : \lambda_1, \dots, \lambda_n \in \mathbb{R}\}$. Similarly, the *affine hull* is the set of all vectors that are affine combinations of the x_i 's and the *convex hull* is the set of all vectors that are convex combinations of the x_i 's.

Consider three linearly independent points in \mathbb{R}^3 . Their linear hull is the entire space. Their affine hull is the plane defined by the points. Their convex hull is the triangle having the

three points as vertices, within the plane defined by the three points. A linear hull always contains the origin ($\lambda_i = 0$).

A set S is *convex* if $\forall x, y \in S$, the line segment from x to y is also contained in S , i.e. any convex combination of x and y is in the set: $\lambda x + (1 - \lambda)y \in S \forall \lambda, 1 \geq \lambda \geq 0$.

One can prove that the convex hull of x_1, \dots, x_m is the smallest convex set containing them.

A *convex polytope* is the convex hull of a finite set of points. It has a sharp, cornered structure. A *hyperplane* is a set of the form $\{x \in \mathbb{R}^n : a \cdot x = t\}$ for some $a \in \mathbb{R}^n$ and $t \in \mathbb{R}$; hence the hyperplane is defined by (a, t) . A *halfspace* is the set of vector that are on one side of a hyperplane: the halfspace determined by (a, t) for $a \in \mathbb{R}^n$ and $t \in \mathbb{R}$ is the set $\{x : a \cdot x \leq t\}$. Are halfspaces convex? Yes, observe that if $a \cdot x \leq t$ and $a \cdot y \leq t$ then $a \cdot (\lambda x + (1 - \lambda)y) \leq t$ for any $0 \leq \lambda \leq 1$. It is also easy to verify that the intersection of two convex sets is convex. A *polyhedron* is the intersection of a finite number of half-spaces.

The following theorem plays a key role in polyhedral theory.

Theorem 2 (Minkowski-Weyl). *Every convex polytope is a polyhedron.*

Definitions

- A *full-dimensional polyhedron* is one that has an interior point (a point that satisfies all the half-spaces inequalities as strict inequalities rather than as equalities).
- The minimal set of half-spaces needed to describe a full-dimensional polytope are its *essential inequalities*. A *facet* is the subset of points of the polyhedron that satisfies an essential inequality as an equality.
- A *vertex* or *extreme point* of a polyhedron is any point that is not a convex combination of 2 other points in the set. It is the unique solution of n linearly independent half spaces, i.e., a point that satisfies n linearly independent essential inequalities as equalities.
- A *face* is a subset (of a polyhedron) of the form $\{x : x \text{ satisfies some subset of the essential inequalities as equalities}\}$. The *dimension* of a face is the dimension of the affine hull of the face. It equals $n - (\text{the number of equations satisfied})$. Thus the dimension of a vertex is 0, of a facet, $n - 1$.

A regular polytope is one in which all vertices have the same degree and every facet has

the same number of edges. We will now prove that there are five regular polytopes (up to symmetry) in three dimensions. The ancient Greeks called them regular solids.

Let f_i be the number of faces of dimension i of a polytope. Then the following identity was proved by Euler:

$$\sum_{i=0}^{n-1} (-1)^i f_i = 1 - (-1)^n$$

So, for $n = 3$, $f_0 - f_1 + f_2 = 2$. Let us check this for a tetrahedron :

$$f_0 = 4 \text{ (vertices)}$$

$$f_1 = 6 \text{ (edges)}$$

$$f_2 = 4 \text{ (facets)}$$

So the relation holds.

Let v be the number of edges at each vertex and e be the number of edges per facet.

1. Now $f_1 = \frac{ef_2}{2}$, as every edge occurs on 2 facets.

2. The number of vertices is $f_0 = \frac{2e}{v}$.

3. From Euler's formula, $f_0 - f_1 + f_2 = 2$.

Together these imply that

$$\frac{e}{v}f_2 - \frac{e}{2}f_2 + f_2 = 2$$

It follows that

$$f_2 = \frac{4v}{2e - (e-2)v}$$

where $e > 3$ since the polytope is in 3 dimensions.

Consider $e = 3$. Then

$$f_2 = \frac{4v}{6-v} \Rightarrow 1 \leq v \leq 5$$

Now $v \geq 3$ and so $v = 3, 4, 5$ are the possibilities. In addition, f_2 must be an integer. So we get $f_2 = 4, 8, 20$ which gives $(4, 6, 4)$, $(6, 12, 8)$ and $(12, 30, 20)$ as the (f_0, f_1, f_2) descriptions of regular polytopes.

Now consider $e = 4$. Then $f_2 = \frac{4v}{8-2v}$ gives $(8, 12, 6)$. No more solutions exist. For $e = 5$, $f_2 = \frac{4v}{10-3v}$ has only $(20, 30, 12)$ as a solution. And there are no solutions for $e \geq 6$. The unique polytopes corresponding to these descriptions are the only 5 regular polytopes in three dimensions.