18.440 PROBLEM SET ONE, DUE FEBRUARY 14

A. FROM ROSS 8th EDITION CHAPTER ONE:

1. **Problem 10:** In how many ways can 8 people be seated in a row if
   (a) there are no restrictions on the seating arrangement?
   (b) persons A and B must sit next to each other?
   (c) there are 4 women and 4 men and no 2 men or 2 women can sit next to each other?
   (d) there are 5 men and they must sit next to each other?
   (e) there are 4 married couples and each couple must sit together?
2. **Problem 26:** Expand $(x_1 + 2x_2 + 3x_3)^4$.
3. **Problem 32:** An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?
4. **Theoretical Exercise 8:** Prove that $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0}$. *Hint:* Consider a group of n men and m women. How many groups of size r are possible?
5. **Theoretical Exercise 11:** The following identity is known as Fermat's combinatorial identity:

   $$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \quad n \geq k.$$

   Give a combinatorial argument (no computations are needed) to establish this identity. *Hint:* Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?
6. **Self-Test Problem/Exercise 17:** Give an analytic verification of

   $$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}, \quad 1 \leq k \leq n$$

   Now give a combinatorial argument for this identity.

B. Suppose you have 12 (distinguishable) hats and 4 (distinguishable) people. How many ways are there to divide the 12 hats among the 4 people with each person getting exactly three hats?

C. Consider permutations $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

1. How many such σ have only one cycle, i.e., have the property that $\sigma(1), \sigma \circ \sigma(1), \sigma \circ \sigma \circ \sigma(1), \ldots$ cycles through all elements of $\{1, 2, \ldots, n\}$?

2. How many σ are fixed-point-free involutions, i.e., have the property that for each j, $\sigma(j) \neq j$ but $\sigma \circ \sigma(j) = j$?

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