18.440 PROBLEM SET SIX DUE APRIL 4

A. FROM TEXTBOOK CHAPTER FIVE:

1. Problem 23: One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that the number 5 will appear less than 150 times.
2. Problem 27: In 10,000 independent tosses of a coin, the coin lands on heads 5800 times. Is it reasonable to assume that the coin is not fair? Explain.
3. Problem 32: The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is
   (a) the probability that a repair time exceeds 2 hours?
   (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?
4. Theoretical Exercise 9: If X is an exponential random variable with parameter λ, and $c > 0$, show that cX is exponential with parameter λ/c.
5. Theoretical Exercise 21: Show that $\Gamma(1/2) = \sqrt{\pi}$. *Hint:* $\Gamma(1/2) = \int_0^\infty e^{-x}x^{-1/2}dx$. Make the change of variables $y = \sqrt{2x}$ and then relate the resulting expression to the normal distribution.
6. Theoretical Exercise 29: Let X be a continuous random variable having cumulative distribution function F. Define the random variable Y by $Y = F(X)$. Show that Y is uniformly distributed over $(0, 1)$.
7. Theoretical Exercise 30: Let X have probability density f_X. Find the probability density function of the random variable Y defined by $Y = aX + b$.

B. At time zero, a single bacterium in a dish divides into two bacteria. This species of bacteria has the following property: after a bacterium B divides into two new bacteria B_1 and B_2, the subsequent length of time until each B_i divides is an exponential random variable of rate $\lambda = 1$, independently of everything else happening in the dish.

1. Compute the expectation of the time T_n at which the number of bacteria reaches n.

2. Compute the variance of T_n.

3. Are both of the answers above unbounded, as functions of n? Give a rough numerical estimate of the values when $n = 10^{50}$.

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