18.440: Lecture 10

Variance and standard deviation

Scott Sheffield

MIT

Outline

Defining variance

Examples

Properties

Decomposition trick

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Recall definitions for expectation

- Recall: a random variable X is a function from the state space to the real numbers.
- Can interpret X as a quantity whose value depends on the outcome of an experiment.
- Say X is a **discrete** random variable if (with probability one) it takes one of a countable set of values.
- For each a in this countable set, write $p(a) := P\{X = a\}$. Call p the **probability mass function**.
- The **expectation** of X, written $E[X]$, is defined by

$$E[X] = \sum_{x:p(x)>0} xp(x).$$

- Also,

$$E[g(X)] = \sum_{x:p(x)>0} g(x)p(x).$$

Defining variance

- Let X be a random variable with mean μ.
- The variance of X, denoted $\mathrm{Var}(X)$, is defined by $\mathrm{Var}(X) = E[(X-\mu)^2]$.
- Taking $g(x) = (x-\mu)^2$, and recalling that $E[g(X)] = \sum_{x:p(x)>0} g(x)p(x)$, we find that

$$\mathrm{Var}[X] = \sum_{x:p(x)>0} (x-\mu)^2 p(x).$$

- Variance is one way to measure the amount a random variable "varies" from its mean over successive trials.

Very important alternate formula

- Let X be a random variable with mean μ.
- We introduced above the formula $\mathrm{Var}(X) = E[(X-\mu)^2]$.
- This can be written $\mathrm{Var}[X] = E[X^2 - 2X\mu + \mu^2]$.
- By additivity of expectation, this is the same as $E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - \mu^2$.
- This gives us our very important alternate formula: $\mathrm{Var}[X] = E[X^2] - (E[X])^2$.
- Original formula gives intuitive idea of what variance is (expected square of difference from mean). But we will often use this alternate formula when we have to actually compute the variance.

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Variance examples

- If X is number on a standard die roll, what is $\mathrm{Var}[X]$?
- $\mathrm{Var}[X] = E[X^2] - E[X]^2 = \frac{1}{6}1^2 + \frac{1}{6}2^2 + \frac{1}{6}3^2 + \frac{1}{6}4^2 + \frac{1}{6}5^2 + \frac{1}{6}6^2 - (7/2)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$.
- Let Y be number of heads in two fair coin tosses. What is $\mathrm{Var}[Y]$?
- Recall $P\{Y = 0\} = 1/4$ and $P\{Y = 1\} = 1/2$ and $P\{Y = 2\} = 1/4$.
- Then $\mathrm{Var}[Y] = E[Y^2] - E[Y]^2 = \frac{1}{4}0^2 + \frac{1}{2}1^2 + \frac{1}{4}2^2 - 1^2 = \frac{1}{2}$.

More variance examples

- You buy a lottery ticket that gives you a one in a million chance to win a million dollars.
- Let X be the amount you win. What's the expectation of X?
- How about the variance?
- Variance is more sensitive than expectation to rare "outlier" events.
- At a particular party, there are four five-foot-tall people, five six-foot-tall people, and one seven-foot-tall person. You pick one of these people uniformly at random. What is the expected height of the person you pick?
- Variance?

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Identity

- If $Y = X + b$, where b is constant, then does it follow that $\mathrm{Var}[Y] = \mathrm{Var}[X]$?
- Yes.
- We showed earlier that $E[aX] = aE[X]$. We claim that $\mathrm{Var}[aX] = a^2\mathrm{Var}[X]$.
- Proof: $\mathrm{Var}[aX] = E[a^2X^2] - E[aX]^2 = a^2E[X^2] - a^2E[X]^2 = a^2\mathrm{Var}[X]$.

Standard deviation

- Write $\mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]}$.
- Satisfies identity $\mathrm{SD}[aX] = a\mathrm{SD}[X]$.
- Uses the same units as X itself.
- If we switch from feet to inches in our "height of randomly chosen person" example, then X, $E[X]$, and $\mathrm{SD}[X]$ each get multiplied by 12, but $\mathrm{Var}[X]$ gets multiplied by 144.

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Number of aces

- Choose five cards from a standard deck of 52 cards. Let A be the number of aces you see.
- Compute $E[A]$ and $\mathrm{Var}[A]$.
- How many five card hands total?
- Answer: $\binom{52}{5}$.
- How many such hands have k aces?
- Answer: $\binom{4}{k}\binom{48}{5-k}$.
- So $P\{A = k\} = \frac{\binom{4}{k}\binom{48}{5-k}}{\binom{52}{5}}$.
- So $E[A] = \sum_{k=0}^{4} kP\{A = k\}$,
- and $\mathrm{Var}[A] = \sum_{k=0}^{4} k^2 P\{A = k\} - E[A]^2$.

Number of aces revisited

- Choose five cards from a standard deck of 52 cards. Let A be the number of aces you see.
- Choose five cards in order, and let A_i be 1 if the ith card chosen is an ace and zero otherwise.
- Then $A = \sum_{i=1}^{5} A_i$. And $E[A] = \sum_{i=1}^{5} E[A_i] = 5/13$.
- Now $A^2 = (A_1 + A_2 + \ldots + A_5)^2$ can be expanded into 25 terms: $A^2 = \sum_{i=1}^{5} \sum_{j=1}^{5} A_i A_j$.
- So $E[A^2] = \sum_{i=1}^{5} \sum_{j=1}^{5} E[A_i A_j]$.
- Five terms of form $E[A_i A_j]$ with $i = j$ five with $i \neq j$. First five contribute $1/13$ each. How about other twenty?
- $E[A_i A_j] = (1/13)(3/51) = (1/13)(1/17)$. So $E[A^2] = \frac{5}{13} + \frac{20}{13\times 17} = \frac{105}{13\times 17}$.
- $\mathrm{Var}[A] = E[A^2] - E[A]^2 = \frac{105}{13\times 17} - \frac{25}{13\times 13}$.

Hat problem variance

- In the n-hat shuffle problem, let X be the number of people who get their own hat. What is $\mathrm{Var}[X]$?
- We showed earlier that $E[X] = 1$. So $\mathrm{Var}[X] = E[X^2] - 1$.
- But how do we compute $E[X^2]$?
- Decomposition trick: write variable as sum of simple variables.
- Let X_i be one if ith person gets own hat and zero otherwise. Then $X = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^n X_i$.
- We want to compute $E[(X_1 + X_2 + \ldots + X_n)^2]$.
- Expand this out and using linearity of expectation:

$$E[\sum_{i=1}^n X_i \sum_{j=1}^n X_j] = \sum_{i=1}^n \sum_{j=1}^n E[X_i X_j] = n \cdot \frac{1}{n} + n(n-1)\frac{1}{n(n-1)} = 2.$$

- So $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 2 - 1 = 1$.