Lecture 8

8.1 Gamma distribution.

Let us take two parameters $\alpha > 0$ and $\beta > 0$. Gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

If we divide both sides by $\Gamma(\alpha)$ we get

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy$$

where we made a change of variables $x = \beta y$. Therefore, if we define

$$f(x|\alpha, \beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

then $f(x|\alpha, \beta)$ will be a probability density function since it is nonnegative and it integrates to one.

Definition. The distribution with p.d.f. $f(x|\alpha, \beta)$ is called Gamma distribution with parameters α and β and it is denoted as $\Gamma(\alpha, \beta)$.

Next, let us recall some properties of gamma function $\Gamma(\alpha)$. If we take $\alpha > 1$ then using integration by parts we can write:

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty x^{\alpha-1} e^{-x} dx = \int_0^\infty x^{\alpha-1} d(-e^{-x}) \\ &= x^{\alpha-1}(-e^{-x})\Big|_0^\infty - \int_0^\infty (-e^{-x})(\alpha-1) x^{\alpha-2} dx \\ &= (\alpha-1) \int_0^\infty x^{(\alpha-1)-1} e^{-x} dx = (\alpha-1)\Gamma(\alpha-1). \end{aligned}$$

Since for $\alpha = 1$ we have

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

we can write

$$\Gamma(2) = 1 \cdot 1,\ \Gamma(3) = 2 \cdot 1,\ \Gamma(4) = 3 \cdot 2 \cdot 1,\ \Gamma(5) = 4 \cdot 3 \cdot 2 \cdot 1$$

and proceeding by induction we get that $\Gamma(n) = (n-1)!$

Let us compute the kth moment of gamma distribution. We have,

$$\begin{aligned}
\mathbb{E}X^k &= \int_0^\infty x^k \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha+k)-1} e^{-\beta x} dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} \underbrace{\int_0^\infty \frac{\beta^{\alpha+k}}{\Gamma(\alpha+k)} x^{\alpha+k-1} e^{-\beta x} dx}_{\textit{p.d.f. of } \Gamma(\alpha+k,\beta) \textit{ integrates to } 1} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k} = \frac{(\alpha+k-1)\Gamma(\alpha+k-1)}{\Gamma(\alpha)\beta^k} \\
&= \frac{(\alpha+k-1)(\alpha+k-2)\ldots\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta^k} = \frac{(\alpha+k-1)\cdots\alpha}{\beta^k}.
\end{aligned}$$

Therefore, the mean is

$$\mathbb{E}X = \frac{\alpha}{\beta}$$

the second moment is

$$\mathbb{E}X^2 = \frac{(\alpha+1)\alpha}{\beta^2}$$

and the variance

$$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{(\alpha+1)\alpha}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}.$$

8.2 Beta distribution.

It is not difficult to show that for $\alpha, \beta > 0$

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Dividing the equation by the right hand side we get that

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = 1$$

which means that the function

$$f(x|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \text{ for } x \in [0,1]$$

is a probability density function. The corresponding distribution is called Beta distribution with parameters α and β and it is denoted as $B(\alpha,\beta)$.

Let us compute the kth moment of Beta distribution.

$$\begin{aligned}
\mathbb{E}X^k &= \int_0^1 x^k \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{k+\alpha-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+k)\Gamma(\beta)}{\Gamma(k+\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \underbrace{\int_0^1 \frac{\Gamma(k+\alpha+\beta)}{\Gamma(\alpha+k)\Gamma(\beta)} x^{(k+\alpha)-1}(1-x)^{\beta-1} dx}_{p.d.f \text{ of } B(k+\alpha,\beta) \text{ integrates to } 1} \\
&= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+k)} = \frac{(\alpha+k-1)(\alpha+k-2)\ldots\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \times \\
&\quad \times \frac{\Gamma(\alpha+\beta)}{(\alpha+\beta+k-1)(\alpha+\beta+k-2)\ldots(\alpha+\beta)\Gamma(\alpha+\beta)} \\
&= \frac{(\alpha+k-1)\ldots\alpha}{(\alpha+\beta+k-1)\ldots(\alpha+\beta)}.
\end{aligned}$$

Therefore, the mean is

$$\mathbb{E}X = \frac{\alpha}{\alpha+\beta}$$

the second moment is

$$\mathbb{E}X^2 = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

and the variance is

$$\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$