18.445 Introduction to Stochastic Processes

Lecture 13: Countable state space chains 2

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Recall Suppose that P is irreducible.

- The Markov chain is recurrent if and only if

$$\mathbb{P}_x[\tau_x^+ < \infty] = 1, \quad \text{for some } x.$$

- The Markov chain is positive recurrent if and only if

$$\mathbb{E}_x[\tau_x^+] < \infty, \quad \text{for some } x.$$

Today's Goal

- stationary distribution
- convergence to stationary distribution

Stationary distribution

Theorem

An irreducible Markov chain is positive recurrent if and only if there exists a probability measure π on Ω such that $\pi = \pi P$.

Corollary

If an irreducible Markov chain is positive recurrent, then

- *there exists a probability measure π such that $\pi = \pi P$;*
- *$\pi(x) > 0$ for all x. In fact,*

$$\pi(x) = \frac{1}{\mathbb{E}_x[\tau_x^+]}.$$

Convergence to the stationary

Theorem

If an irreducible Markov chain is positive recurrent and aperiodic, then

$$\lim_n \mathbb{P}_x[X_n = y] = \pi(y) > 0, \quad \textit{for all } x, y.$$

Theorem

If an irreducible Markov chain is null recurrent, then

$$\lim_n \mathbb{P}_x[X_n = y] = 0, \quad \textit{for all } x, y.$$

Convergence to the stationary

Recall Consider a Markov chain with state space Ω (countable) and transition matrix P. For each $x \in \Omega$, define

$$T(x) = \{n \geq 1 : P^n(x, x) > 0\}.$$

Then

$$gcd(T(x)) = gcd(T(y)), \quad \text{for all } x, y.$$

We say the chain is aperiodic if $gcd(T(x)) = 1$.

Theorem

Suppose that the Markov chain is irreducible and aperiodic. If the chain is positive recurrent, then

$$\lim_n ||P^n(x, \cdot) - \pi||_{TV} = 0.$$