18.440 Midterm 1, Spring 2014: 50 minutes, 100 points

1. (10 points) How many quintuples $(a_1, a_2, a_3, a_4, a_5)$ of non-negative integers satisfy $a_1 + a_2 + a_3 + a_4 + a_5 = 100$? **ANSWER:** This the number of ways to make a list of "100 stars and 4 bars", which is $\binom{104}{4} = \frac{104!}{4!100!}$.

2. (20 points) Thirty people are invited to a party. Each person accepts the invitation, independently of all others, with probability 1/3. Let X be the number of accepted invitations. Compute the following:

(a) $E[X]$ **ANSWER:** X is binomial with $n = 30$ and $p = 1/3$, so the expectation is $np = 10$.

(b) $\text{Var}[X]$ **ANSWER:** X is binomial with $n = 30$ and $p = 1/3$, so the variance is $npq = np(1-p) = 20/3$.

(c) $E[X^2]$ **ANSWER:** $\text{Var}(X) = E[X^2] - E[X]^2$. Using previous two parts and solving gives $E[X^2] = 20/3 + 100 = 320/3$.

(d) $E[X^2 - 4X + 5]$ **ANSWER:** By linearity of expectation, this is $E[X^2] - 4E[X] + 5 = 320/3 - 40 + 5 = 215/3$.

3. (20 points) Bob has noticed that during every given minute, there is a 1/720 chance that the Facebook page for his dry cleaning business will get a "like", independently of what happens during any other minute. Let L be the total number of likes that Bob receives during a 24 hour period.

(a) Compute $E[L]$ and $\text{Var}[L]$. (Give exact answers, not approximate ones.) **ANSWER:** This is binomial with $n = 60 \times 24$ and $p = 1/720$. So $E[L] = np = 2$ and $\text{Var}[L] = np(1-p) = 2\frac{719}{720}$.

(b) Compute the probability that $L = 0$. (Give an exact answer, not an approximate answer.) **ANSWER:** $(1-p)^n = \left(\frac{719}{720}\right)^{1440}$

(c) Bob is really hoping to get at least 2 more likes during the next 24 hours (because this would boost his cumulative total to triple digits). Use a Poisson random variable calculation to *approximate* the probability that $L \geq 2$. **ANSWER:** Note that L is approximately binomial with parameter $\lambda = E[L] = 2$. Thus $P\{L \geq 2\} = 1 - P\{L = 1\} - P\{L = 0\} \approx 1 - e^{-\lambda}\lambda^0/0! - e^{-\lambda}\lambda^1/1! = 1 - 3e^{-2} = 1 - 3/e^2$

4. (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p. Compute (in terms of p) the

probability that the fifth head occurs on the tenth toss. **ANSWER:** This is the probability that exactly four of the first nine tosses are heads, and then the tenth toss is also heads. This comes to $\binom{9}{4}p^5(1-p)^5$.

5. (20 points) Let X be the number on a standard die roll (assuming values in $\{1,2,3,4,5,6\}$ with equal probability). Let Y be the number on an independent roll of the same die. Compute the following:

(a) The expectation $E[X^2]$. **ANSWER:** $(1+4+9+16+25+36)/6 = 91/6$.

(b) The expectation $E[XY]$. **ANSWER:** By independence of X and Y, we have $E[XY] = E[X]E[Y] = (7/2)^2 = 49/4$.

(c) The covariance $\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$. **ANSWER:** Because of independence, $\text{Cov}(X,Y) = 0$.

6. (20 points) Three hats fall out of their assigned bins and are randomly placed back in bins, one hat per bin (with all 3! reassignments being equally likely). Compute the following:

(a) The expected number of hats that end up in their own bins. **ANSWER:** Let X_i be 1 if ith hat ends up in own bin, zero otherwise. Then $X = X_1 + X_2 + X_3$ is total number of hats to end up in their own bins, and $E[X] = E[X_1] + E[X_2] + E[X_3] = 3\frac{1}{3} = 1$.

(b) The probability that the third hat ends up in its own bin. **ANSWER:** 1/3

(b) The conditional probability that the third hat ends up in its own bin *given* that the first hat does *not* end up in its own bin. **ANSWER:** Let A be event third hat gets own bin, B event that first hat does not end up in its own bin. Then $P(A) = 1/3$ and $P(B) = 2/3$. There is only one permutation that assigns the third hat to its own bin and does not assign first hat to its own bin, so $P(AB) = 1/6$. Thus $P(A|B) = P(AB)/P(B) = (1/6)/(2/3) = 1/4$.