18.600: Lecture 18

Normal random variables

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Outline

Tossing coins

Normal random variables

Special case of central limit theorem

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- Let's try this out.

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- Then switch to polar coordinates.

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-r^2/2} r d\theta dr = 2\pi \int_0^{\infty} r e^{-r^2/2} dr = -2\pi e^{-r^2/2} \Big|_0^{\infty},$$

so $I = \sqrt{2\pi}$.

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- How would we compute $\mathrm{Var}[X] = \int f(x)x^2 dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^2 dx$?
- Try integration by parts with $u = x$ and $dv = xe^{-x^2/2}dx$. Find that $\mathrm{Var}[X] = \frac{1}{\sqrt{2\pi}} (-xe^{-x^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx) = 1$.

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- $E[Y] = E[X] + \mu = \mu$ and $\mathrm{Var}[Y] = \sigma^2 \mathrm{Var}[X] = \sigma^2$.

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- Values: $\Phi(-3) \approx .0013$, $\Phi(-2) \approx .023$ and $\Phi(-1) \approx .159$.
- Rough rule of thumb: "two thirds of time within one SD of mean, 95 percent of time within 2 SDs of mean."

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- This is $\Phi(b) - \Phi(a) = P\{a \le X \le b\}$ when X is a standard normal random variable.

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- Here $\sqrt{npq} = \sqrt{60000 \times \frac{1}{6} \times \frac{5}{6}} \approx 91.28$.
- And $200/91.28 \approx 2.19$. Answer is about $1 - \Phi(-2.19)$.

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