18.600: Lecture 2

Multinomial coefficients and more counting problems

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Outline

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Multinomial coefficients

Integer partitions

More problems

Partition problems

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- One way to think of this: given any permutation of eight elements (e.g., 12435876 or 87625431) declare first three as breakfast, second two as lunch, last three as dinner. This maps set of $8!$ permutations on to the set of food-meal divisions in a many-to-one way: each food-meal division comes from $3!2!3!$ permutations.

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- How many 8-letter sequences with 3 A's, 2 B's, and 3 C's?
- Answer: $8!/(3!2!3!)$. Same as other problem. Imagine 8 "slots" for the letters. Choose 3 to be A's, 2 to be B's, and 3 to be C's.

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- Answer $\binom{n}{n_1, n_2, \dots, n_r} := \frac{n!}{n_1! n_2! \dots n_r!}$.

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- Generally, $(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$, because there are $\binom{n}{k}$ sequences with k A's and $(n - k)$ B's.

How about trinomials?

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- What is the sum of the coefficients in this expansion? What is the combinatorial interpretation of coefficient of, say, ABC^2?
- Answer $81 = (1 + 1 + 1)^4$. ABC^2 has coefficient 12 because there are 12 length-4 words have one A, one B, two C's.

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- Yes (look it up) but it is a bit tricker to draw and visualize than Pascal's triangle.

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- **Because** we want the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to still make sense when $k = 0$ and $k = n$. There is clearly 1 way to choose n elements from a group of n elements. And 1 way to choose 0 elements from a group of n elements so $\frac{n!}{n!0!} = \frac{n!}{0!n!} = 1$.

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- Another common notation: Write $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$ and define $n! := \Gamma(n+1) = \int_0^\infty t^n e^{-t} dt$, so that $\Gamma(n) = (n-1)!$.

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- Answer: $\binom{n+k-1}{n}$. Represent partition by $k-1$ bars and n stars, e.g., as $** \mid **|| ****|*$.

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- $\binom{179}{90} = \binom{179}{89}$

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- Need three 3-card suits, one 4-card suit, to make 13 cards total. Answer is $4\binom{13}{3}^3\binom{13}{4}$

18.600 Probability and Random Variables
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