18.600: Lecture 35

Martingales and risk neutral probability

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MIT

Outline

Martingales and stopping times

Martingales and Bayesian expectation revisions

Risk neutral probability and martingales

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Martingales and Bayesian expectation revisions

Risk neutral probability and martingales

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- "Given all I know today, expected price tomorrow is the price today."

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- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.
- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

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- What is the probability that it goes down to 45 then up to 55 then down to 45 then up to 55 again — all before reaching either 0 or 100?

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- More generally, $E[X|\mathcal{F}_0], E[X|\mathcal{F}_1], E[X|\mathcal{F}_2], \ldots$ is a martingale,

Martingales as sequentially updated probability estimates

- Example: let C be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of C. Let C_n be the **conditional expectation** of C *given* the outcome of the first n of these tests. Then the sequence $C_0, C_1, C_2, \ldots, C_{10} = C$ is a martingale.

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- This is *not* a statement about how well informed my probability measure is.

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- Call me!!! I love you! Alice 0

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- But there are some caveats: interest, risk premium, etc.
- According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respected to **risk neutral probability**.

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- Risk neutral probability is the probability determined by the market betting odds.

Risk neutral probability of outcomes known at fixed time T

- **Risk neutral probability of event A:** $P_{RN}(A)$ denotes

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- Assuming no **arbitrage** (i.e., no risk free profit with zero upfront investment), P_{RN} satisfies axioms of probability. That is, $0 \leq P_{RN}(A) \leq 1$, and $P_{RN}(S) = 1$, and if events A_j are disjoint then $P_{RN}(A_1 \cup A_2 \cup \ldots) = P_{RN}(A_1) + P_{RN}(A_2) + \ldots$

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- **Arbitrage example:** if A and B are disjoint and $P_{RN}(A \cup B) < P(A) + P(B)$ then we sell contracts paying 1 if A occurs and 1 if B occurs, buy contract paying 1 if $A \cup B$ occurs, pocket difference.

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- Now, suppose there are only 2 outcomes: A is event that economy booms and everyone prospers and B is event that economy sags and everyone is needy. Suppose purchasing power of dollar is the same in both scenarios. If people think A has a .5 chance to occur, do we expect $P_{RN}(A) > .5$ or $P_{RN}(A) < .5$?

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- Answer: $P_{RN}(A) < .5$. People are risk averse. In second scenario they need the money more.

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- Even if some people bet based on loyalty, emotion, insurance against personal financial exposure to team's prospects, etc., there will arguably be enough in-it-for-the-money statistical arbitrageurs to keep price near a reasonable guess of what well-informed informed experts would consider the true probability.

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- Risk neutral probability can be defined for variable times and variable interest rates — e.g., one can take the numéraire to be amount one dollar in a variable-interest-rate money market account has grown to when outcome is known. Can define $P_{RN}(A)$ to be price of contract paying this amount if and when A occurs.

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- For simplicity, we focus on fixed time T, fixed interest rate r in this lecture.

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- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
- Answer: $(2P_{RN}(A) + 3P_{RN}(B))e^{-rT}$.

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- If r is risk free interest rate, then by definition, price of a contract paying dollar at time T if A occurs is $P_{RN}(A)e^{-rT}$.
- If A and B are disjoint, what is the price of a contract that pays 2 dollars if A occurs, 3 if B occurs, 0 otherwise?
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- Generally, in absence of arbitrage, price of contract that pays X at time T should be $E_{RN}(X)e^{-rT}$ where E_{RN} denotes expectation with respect to the risk neutral probability.

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- Implies **fundamental theorem of asset pricing**, which says discounted price $\frac{X(n)}{A(n)}$ (where A is a risk-free asset) is a martingale with respected to **risk neutral probability**.

18.600 Probability and Random Variables
Fall 2019