6 Introduction to categories

6.1 The definition of a category

We have now seen many examples of representation theories and of operations with representations (direct sum, tensor product, induction, restriction, reflection functors, etc.) A context in which one can systematically talk about this is provided by **Category Theory**.

Category theory was founded by Saunders MacLane and Samuel Eilenberg around 1940. It is a fairly abstract theory which seemingly has no content, for which reason it was christened "abstract nonsense". Nevertheless, it is a very flexible and powerful language, which has become totally indispensable in many areas of mathematics, such as algebraic geometry, topology, representation theory, and many others.

We will now give a very short introduction to Category theory, highlighting its relevance to the topics in representation theory we have discussed. For a serious acquaintance with category theory, the reader should use the classical book [McL].

Definition 6.1. A category $\mathcal{C}$ is the following data:

(i) a class of objects $Ob(\mathcal{C})$;

(ii) for every objects $X, Y \in Ob(\mathcal{C})$, the class $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Hom}(X, Y)$ of morphisms (or arrows) from X, Y (for $f \in \mathrm{Hom}(X, Y)$, one may write $f : X \to Y$);

(iii) For any objects $X, Y, Z \in Ob(\mathcal{C})$, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(f, g) \mapsto f \circ g$,

which satisfy the following axioms:

1. The composition is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$;

2. For each $X \in Ob(\mathcal{C})$, there is a morphism $1_X \in \mathrm{Hom}(X, X)$, called the unit morphism, such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any f, g for which compositions make sense.

   Remark. We will write $X \in \mathcal{C}$ instead of $X \in Ob(\mathcal{C})$.

Example 6.2. 1. The category **Sets** of sets (morphisms are arbitrary maps).

2. The categories **Groups**, **Rings** (morphisms are homomorphisms).

3. The category **Vect**$_k$ of vector spaces over a field k (morphisms are linear maps).

4. The category $\mathrm{Rep}(A)$ of representations of an algebra A (morphisms are homomorphisms of representations).

5. The category of topological spaces (morphisms are continuous maps).

6. The homotopy category of topological spaces (morphisms are homotopy classes of continuous maps).

   Important remark. Unfortunately, one cannot simplify this definition by replacing the word "class" by the much more familiar word "set". Indeed, this would rule out the important Example 1, as it is well known that there is no set of all sets, and working with such a set leads to contradictions. The precise definition of a class and the precise distinction between a class and a set is the subject of set theory, and cannot be discussed here. Luckily, for most practical purposes (in particular, in these notes), this distinction is not essential.

We also mention that in many examples, including examples 1-6, the word "class" in (ii) can be replaced by "set". Categories with this property (that $\text{Hom}(X,Y)$ is a set for any X,Y) are called locally small; many categories that we encounter are of this kind.

Sometimes the collection $\text{Hom}(X,Y)$ of morphisms from X to Y in a given locally small category $\mathcal{C}$ is not just a set but has some additional structure (say, the structure of an abelian group, or a vector space over some field). In this case one says that $\mathcal{C}$ is **enriched** over another category $\mathcal{D}$ (which is a *monoidal* category, i.e., has a product operation and a unit object under this product, e.g. the category of abelian groups or vector spaces with the tensor product operation). This means that for each $X,Y \in \mathcal{C}$, $\text{Hom}(X,Y)$ is an object of $\mathcal{D}$, and the composition $\text{Hom}(Y,Z) \times \text{Hom}(X,Y) \to \text{Hom}(X,Z)$ is a morphism in $\mathcal{D}$. E.g., if $\mathcal{D}$ is the category of vector spaces, this means that the composition is bilinear, i.e. gives rise to a linear map $\text{Hom}(Y,Z) \otimes \text{Hom}(X,Y) \to \text{Hom}(X,Z)$. For a more detailed discussion of this, we refer the reader to [McL].

Example. The category $\text{Rep}(A)$ of representations of a k-algebra A is enriched over the category of k-vector spaces.

Definition 6.3. A full subcategory of a category $\mathcal{C}$ is a category $\mathcal{C}'$ whose objects are a subclass of objects of $\mathcal{C}$, and $\text{Hom}_{\mathcal{C}'}(X,Y) = \text{Hom}_{\mathcal{C}}(X,Y)$.

Example. The category **AbelianGroups** is a full subcategory of the category **Groups**.

6.2 Functors

We would like to define arrows between categories. Such arrows are called **functors**.

Definition 6.4. A functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is

(i) a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$;

(ii) for each $X,Y \in \mathcal{C}$, a map $F = F_{X,Y} : \text{Hom}(X,Y) \to \text{Hom}(F(X),F(Y))$ which preserves compositions and identity morphisms.

Note that functors can be composed in an obvious way. Also, any category has the identity functor.

Example 6.5. 1. A (locally small) category $\mathcal{C}$ with one object X is the same thing as a monoid. A functor between such categories is a homomorphism of monoids.

2. Forgetful functors **Groups** $\to$ **Sets**, **Rings** $\to$ **AbelianGroups**.

3. The opposite category of a given category is the same category with the order of arrows and compositions reversed. Then $V \mapsto V^*$ is a functor $\textbf{Vect}_k \mapsto \textbf{Vect}_k^{op}$.

4. The Hom functors: If $\mathcal{C}$ is a locally small category then we have the functor $\mathcal{C} \to$ **Sets** given by $Y \mapsto \text{Hom}(X,Y)$ and $\mathcal{C}^{op} \to$ **Sets** given by $Y \mapsto \text{Hom}(Y,X)$.

5. The assignment $X \mapsto \text{Fun}(X,\mathbb{Z})$ is a functor **Sets** $\to$ **Rings**^{op}.

6. Let Q be a quiver. Consider the category $\mathcal{C}(Q)$ whose objects are the vertices and morphisms are oriented paths between them. Then functors from $\mathcal{C}(Q)$ to $\textbf{Vect}_k$ are representations of Q over k.

7. Let $K \subset G$ be groups. Then we have the induction functor $\text{Ind}_K^G : \text{Rep}(K) \to \text{Rep}(G)$, and $\text{Res}_K^G : \text{Rep}(G) \to \text{Rep}(K)$.

8. We have an obvious notion of the Cartesian product of categories (obtained by taking the Cartesian products of the classes of objects and morphisms of the factors). The functors of direct sum and tensor product are then functors $\mathbf{Vect}_k \times \mathbf{Vect}_k \to \mathbf{Vect}_k$. Also the operations $V \mapsto V^{\otimes n}$, $V \mapsto S^n V$, $V \mapsto \wedge^n V$ are functors on $\mathbf{Vect}_k$. More generally, if π is a representation of S_n, we have functors $V \mapsto \mathrm{Hom}_{S_n}(\pi, V^{\otimes n})$. Such functors (for irreducible π) are called the Schur functors. They are labeled by Young diagrams.

9. The reflection functors $F_i^\pm : \mathrm{Rep}(Q) \to \mathrm{Rep}(\bar{Q}_i)$ are functors between representation categories of quivers.

6.3   Morphisms of functors

One of the important features of functors between categories which distinguishes them from usual maps or functions is that the functors between two given categories themselves form a category, i.e., one can define a nontrivial notion of a morphism between two functors.

Definition 6.6. Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ be functors between them. A morphism $a : F \to G$ (also called a natural transformation or a functorial morphism) is a collection of morphisms $a_X : F(X) \to G(X)$ labeled by the objects X of $\mathcal{C}$, which is *functorial* in X, i.e., for any morphism $f : X \to Y$ (for $X, Y \in \mathcal{C}$) one has $a_Y \circ F(f) = G(f) \circ a_X$.

A morphism $a : F \to G$ is an isomorphism if there is another morphism $a^{-1} : G \to F$ such that $a \circ a^{-1}$ and $a^{-1} \circ a$ are the identities. The set of morphisms from F to G is denoted by $\mathrm{Hom}(F, G)$.

Example 6.7. 1. Let $\mathbf{FVect}_k$ be the category of finite dimensional vector spaces over k. Then the functors id and $**$ on this category are isomorphic. The isomorphism is defined by the standard maps $a_V : V \to V^{**}$ given by $a_V(u)(f) = f(u)$, $u \in V$, $f \in V^*$. But these two functors are not isomorphic on the category of all vector spaces $\mathbf{Vect}_k$, since for an infinite dimensional vector space V, V is not isomorphic to V^{**}.

2. Let $\mathbf{FVect}_k'$ be the category of finite dimensional k-vector spaces, where the morphisms are the isomorphisms. We have a functor F from this category to itself sending any space V to V^* and any morphism a to $(a^*)^{-1}$. This functor satisfies the property that V is isomorphic to $F(V)$ for any V, but it is not isomorphic to the identity functor. This is because the isomorphism $V \to F(V) = V^*$ cannot be chosen to be compatible with the action of $GL(V)$, as V is not isomorphic to V^* as a representation of $GL(V)$.

3. Let A be an algebra over a field k, and $F : A - \mathbf{mod} \to \mathbf{Vect}_k$ be the forgetful functor. Then as follows from Problem 1.22, $\mathrm{End}F = \mathrm{Hom}(F, F) = A$.

4. The set of endomorphisms of the identity functor on the category $A - \mathbf{mod}$ is the center of A (check it!).

6.4   Equivalence of categories

When two algebraic or geometric objects are isomorphic, it is usually not a good idea to say that they are equal (i.e., literally the same). The reason is that such objects are usually equal in many different ways, i.e., there are many ways to pick an isomorphism, but by saying that the objects are equal we are misleading the reader or listener into thinking that we are providing a certain choice of the identification, which we actually do not do. A vivid example of this is a finite dimensional vector space V and its dual space V^*.

For this reason in category theory, one most of the time tries to avoid saying that two objects or two functors are equal. In particular, this applies to the definition of isomorphism of categories.

Namely, the naive notion of isomorphism of categories is defined in the obvious way: a functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism if there exists $F^{-1} : \mathcal{D} \to \mathcal{C}$ such that $F \circ F^{-1}$ and $F^{-1} \circ F$ are equal to the identity functors. But this definition is not very useful. We might suspect so since we have used the word "equal" for objects of a category (namely, functors) which we are not supposed to do. And in fact here is an example of two categories which are "the same for all practical purposes" but are not isomorphic; it demonstrates the deficiency of our definition.

Namely, let $\mathcal{C}_1$ be the simplest possible category: $Ob(\mathcal{C}_1)$ consists of one object X, with $\mathrm{Hom}(X, X) = \{1_X\}$. Also, let $\mathcal{C}_2$ have two objects X, Y and 4 morphisms: $1_X, 1_Y, a : X \to Y$ and $b : Y \to X$. So we must have $a \circ b = 1_Y$, $b \circ a = 1_X$.

It is easy to check that for any category $\mathcal{D}$, there is a natural bijection between the collections of isomorphism classes of functors $\mathcal{C}_1 \to \mathcal{D}$ and $\mathcal{C}_2 \to \mathcal{D}$ (both are identified with the collection of isomorphism classes of objects of $\mathcal{D}$). This is what we mean by saying that $\mathcal{C}_1$ and $\mathcal{C}_2$ are "the same for all practical purposes". Nevertheless they are not isomorphic, since $\mathcal{C}_1$ has one object, and $\mathcal{C}_2$ has two objects (even though these two objects are isomorphic to each other).

This shows that we should adopt a more flexible and less restrictive notion of isomorphism of categories. This is accomplished by the definition of an **equivalence of categories.**

Definition 6.8. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists $F' : \mathcal{D} \to \mathcal{C}$ such that $F \circ F'$ and $F' \circ F$ are **isomorphic** to the identity functors.

In this situation, F' is said to be *a quasi-inverse* to F.

In particular, the above categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent (check it!).

Also, the category **FSet** of finite sets is equivalent to the category whose objects are nonnegative integers, and morphisms are given by $\mathrm{Hom}(m, n) = \mathrm{Maps}(\{1, ..., m\}, \{1, ..., n\})$. Are these categories isomorphic? The answer to this question depends on whether you believe that there is only one finite set with a given number of elements, or that there are many of those. It seems better to think that there are many (without asking "how many"), so that isomorphic sets need not be literally equal, but this is really a matter of choice. In any case, this is not really a reasonable question; the answer to this question is irrelevant for any practical purpose, and thinking about it will give you nothing but a headache.

6.5   Representable functors

A fundamental notion in category theory is that of a **representable functor**. Namely, let $\mathcal{C}$ be a (locally small) category, and $F : \mathcal{C} \to \mathbf{Sets}$ be a functor. We say that F is **representable** if there exists an object $X \in \mathcal{C}$ such that F is isomorphic to the functor $\mathrm{Hom}(X, ?)$. More precisely, if we are given such an object X, together with an isomorphism $\xi : F \cong \mathrm{Hom}(X, ?)$, we say that the functor F is **represented by** X (using ξ).

In a similar way, one can talk about representable functors from $\mathcal{C}^{op}$ to **Sets**. Namely, one calls such a functor representable if it is of the form $\mathrm{Hom}(?, X)$ for some object $X \in \mathcal{C}$, up to an isomorphism.

Not every functor is representable, but if a representing object X exists, then it is unique. Namely, we have the following lemma.

Lemma 6.9. *(The Yoneda Lemma) If a functor F is represented by an object X, then X is unique up to a unique isomorphism. I.e., if X, Y are two objects in $\mathcal{C}$, then for any isomorphism of functors $\phi : \mathrm{Hom}(X, ?) \to \mathrm{Hom}(Y, ?)$ there is a unique isomorphism $a_\phi : X \to Y$ inducing ϕ.*

Proof. (Sketch) One sets $a_\phi = \phi_Y^{-1}(1_Y)$, and shows that it is invertible by constructing the inverse, which is $a_\phi^{-1} = \phi_X(1_X)$. It remains to show that the composition both ways is the identity, which we will omit here. This establishes the existence of a_ϕ. Its uniqueness is verified in a straightforward manner. $\qquad\square$

Remark. In a similar way, if a category $\mathcal{C}$ is enriched over another category $\mathcal{D}$ (say, the category of abelian groups or vector spaces), one can define the notion of a representable functor from $\mathcal{C}$ to $\mathcal{D}$.

Example 6.10. Let A be an algebra. Then the forgetful functor to vector spaces on the category of left A-modules is representable, and the representing object is the free rank 1 module (=the regular representation) $M = A$. But if A is infinite dimensional, and we restrict attention to the category of finite dimensional modules, then the forgetful functor, in general, is not representable (this is so, for example, if A is the algebra of complex functions on $\mathbb{Z}$ which are zero at all points but finitely many).

6.6 Adjoint functors

Another fundamental notion in category theory is the notion of **adjoint functors**.

Definition 6.11. Functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are said to be a pair of adjoint functors if for any $X \in \mathcal{C}, Y \in \mathcal{D}$ we are given an isomorphism $\xi_{XY} : \mathrm{Hom}_\mathcal{C}(F(X), Y) \to \mathrm{Hom}_\mathcal{D}(X, G(Y))$ which is functorial in X and Y; in other words, if we are given an isomorphism of functors $\mathrm{Hom}(F(?), ?) \to \mathrm{Hom}(?, G(?))$ ($\mathcal{C} \times \mathcal{D} \to \mathbf{Sets}$). In this situation, we say that F is left adjoint to G and G is right adjoint to F.

Not every functor has a left or right adjoint, but if it does, it is unique and can be constructed canonically (i.e., if we somehow found two such functors, then there is a canonical isomorphism between them). This follows easily from the Yoneda lemma, as if F, G are a pair of adjoint functors then $F(X)$ represents the functor $Y \mapsto \mathrm{Hom}(X, G(Y))$, and $G(Y)$ represents the functor $X \mapsto \mathrm{Hom}(F(X), Y)$.

Remark 6.12. The terminology "left and right adjoint functors" is motivated by the analogy between categories and inner product spaces. More specifically, we have the following useful dictionary between category theory and linear algebra, which helps understand better many notions of category theory.

Dictionary between category theory and linear algebra	
Category $\mathcal{C}$	Vector space V with a nondegenerate inner product
The set of morphisms $\mathrm{Hom}(X, Y)$	Inner product (x, y) on V (maybe nonsymmetric)
Opposite category $\mathcal{C}^{op}$	Same space V with reversed inner product
The category **Sets**	The ground field k
Full subcategory in $\mathcal{C}$	Nondegenerate subspace in V
Functor $F : \mathcal{C} \to \mathcal{D}$	Linear operator $f : V \to W$
Functor $F : \mathcal{C} \to$ **Sets**	Linear functional $f \in V^* = \mathrm{Hom}(V, k)$
Representable functor	Linear functional $f \in V^*$ given by $f(v) = (u, v)$, $u \in V$
Yoneda lemma	Nondegeneracy of the inner product (on both sides)
Not all functors are representable	If $\dim V = \infty$, not $\forall f \in V^*$, $f(v) = (u, v)$
Left and right adjoint functors	Left and right adjoint operators
Adjoint functors don't always exist	Adjoint operators may not exist if $\dim V = \infty$
If they do, they are unique	If they do, they are unique
Left and right adjoints may not coincide	The inner product may be nonsymmetric

Example 6.13. 1. Let V be a finite dimensional representation of a group G or a Lie algebra $\mathfrak{g}$. Then the left and right adjoint to the functor $V \otimes$ on the category of representations of G is the functor $V^* \otimes$.

2. The functor Res_K^G is left adjoint to Ind_K^G. This is nothing but the statement of the Frobenius reciprocity.

3. Let $\mathbf{Assoc}_k$ be the category of associative unital algebras, and $\mathbf{Lie}_k$ the category of Lie algebras over some field k. We have a functor $L : \mathbf{Assoc}_k \to \mathbf{Lie}_k$, which attaches to an associative algebra the same space regarded as a Lie algebra, with bracket $[a, b] = ab - ba$. Then the functor L has a left adjoint, which is the functor U of taking the universal enveloping algebra of a Lie algebra.

4. We have the functor $GL_1 : \mathbf{Assoc}_k \to \mathbf{Groups}$, given by $A \mapsto GL_1(A) = A^\times$. This functor has a left adjoint, which is the functor $G \mapsto k[G]$, the group algebra of G.

5. The left adjoint to the forgetful functor $\mathbf{Assoc}_k \to \mathbf{Vect}_k$ is the functor of tensor algebra: $V \mapsto TV$. Also, if we denote by $\mathbf{Comm}_k$ the category of commutative algebras, then the left adjoint to the forgetful functor $\mathbf{Comm}_k \to \mathbf{Vect}_k$ is the functor of the symmetric algebra: $V \mapsto SV$.

One can give many more examples, spanning many fields. These examples show that adjoint functors are ubiquitous in mathematics.

6.7 Abelian categories

The type of categories that most often appears in representation theory is **abelian categories**. The standard definition of an abelian category is rather long, so we will not give it here, referring the reader to the textbook [Fr]; rather, we will use as the definition what is really the statement of the Freyd-Mitchell theorem:

Definition 6.14. An abelian category is a category (enriched over the category of abelian groups), which is equivalent to a full subcategory $\mathcal{C}$ of the category A-mod of left modules over a ring A, closed under taking finite direct sums, as well as kernels, cokernels, and images of morphisms.

We see from this definition that in an abelian category, $\mathrm{Hom}(X, Y)$ is an abelian group for each X, Y, compositions are group homomorphisms with respect to each argument, there is the zero object, the notion of an injective morphism (monomorphism) and surjective morphism (epimorphism), and every morphism has a kernel, a cokernel, and an image.

Example 6.15. The category of modules over an algebra A and the category of finite dimensional modules over A are abelian categories.

Remark 6.16. The good thing about Definition 6.14 is that it allows us to visualize objects, morphisms, kernels, and cokernels in terms of classical algebra. But the definition also has a big drawback, which is that even if $\mathcal{C}$ is the whole category A-mod, the ring A is not determined by $\mathcal{C}$. In particular, two different rings can have equivalent categories of modules (such rings are called **Morita equivalent**). Actually, it is worse than that: for many important abelian categories there is no natural (or even manageable) ring A at all. This is why people prefer to use the standard definition, which is free from this drawback, even though it is more abstract.

We say that an abelian category $\mathcal{C}$ is k-**linear** if the groups $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are equipped with a structure of a vector space over k, and composition maps are k-linear in each argument. In particular, the categories in Example 6.15 are k-linear.

6.8 Exact functors

Definition 6.17. A sequence of objects and morphisms

$$X_0 \to X_1 \to ... \to X_{n+1}$$

in an abelian category is said to be **a complex** if the composition of any two consecutive arrows is zero. The **cohomology** of this complex is $H^i = \mathrm{Ker}\,(d_i)/\mathrm{Im}(d_{i-1})$, where $d_i : X_i \to X_{i+1}$ (thus the cohomology is defined for $1 \le i \le n$). The complex is said to be **exact in the i-th term** if $H^i = 0$, and is said to be **an exact sequence** if it is exact in all terms. A **short exact sequence** is an exact sequence of the form

$$0 \to X \to Y \to Z \to 0.$$

Clearly, $0 \to X \to Y \to Z \to 0$ is a short exact sequence if and only if $X \to Y$ is injective, $Y \to Z$ is surjective, and the induced map $Y/X \to Z$ is an isomorphism.

Definition 6.18. A functor F between two abelian categories is **additive** if it induces homomorphisms on Hom groups. Also, for k-linear categories one says that F is k-linear if it induces k-linear maps between Hom spaces.

It is easy to show that if F is an additive functor, then $F(X \oplus Y)$ is canonically isomorphic to $F(X) \oplus F(Y)$.

Example 6.19. The functors Ind_K^G, Res_K^G, $\mathrm{Hom}_G(V, ?)$ in the theory of group representations over a field k are additive and k-linear.

Definition 6.20. An additive functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is **left exact** if for any exact sequence

$$0 \to X \to Y \to Z,$$

the sequence

$$0 \to F(X) \to F(Y) \to F(Z)$$

is exact. F is **right exact** if for any exact sequence

$$X \to Y \to Z \to 0,$$

the sequence

$$F(X) \to F(Y) \to F(Z) \to 0$$

is exact. F is **exact** if it is both left and right exact.

Definition 6.21. An abelian category $\mathcal{C}$ is **semisimple** if any short exact sequence in this category splits, i.e., is isomorphic to a sequence

$$0 \to X \to X \oplus Y \to Y \to 0$$

(where the maps are obvious).

Example 6.22. The category of representations of a finite group G over a field of characteristic not dividing $|G|$ (or 0) is semisimple.

Note that in a semisimple category, any additive functor is automatically exact on both sides.

Example 6.23. (i) The functors Ind_K^G, Res_K^G are exact.

(ii) The functor $\mathrm{Hom}(X, ?)$ is left exact, but not necessarily right exact. To see that it need not be right exact, it suffices to consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

and apply the functor $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, ?)$.

(iii) The functor $X \otimes_A$ for a right A-module X (on the category of left A-modules) is right exact, but not necessarily left exact. To see this, it suffices to tensor multiply the above exact sequence by $\mathbb{Z}/2\mathbb{Z}$.

Exercise. Show that if (F, G) is a pair of adjoint additive functors between abelian categories, then F is right exact and G is left exact.

Exercise. (a) Let Q be a quiver and $i \in Q$ a source. Let V be a representation of Q, and W a representation of $\overline{Q_i}$ (the quiver obtained from Q by reversing arrows at the vertex i). Prove that there is a natural isomorphism between $\mathrm{Hom}\left(F_i^- V, W\right)$ and $\mathrm{Hom}\left(V, F_i^+ W\right)$. In other words, the functor F_i^+ is right adjoint to F_i^-.

(b) Deduce that the functor F_i^+ is left exact, and F_i^- is right exact.

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