October 16, 2003.

1 Recap

We know now that products of prevarieties exist.

Here's a note about the next homework:

Let $\mathcal{C}$ be a category, let $X \in \mathcal{C}$. Then we define $h_X : C \to Set$ defined by $h_X : Y \mapsto \mathrm{Hom}_C(Y, X)$.

Yoneda's Lemma. If $X, Y \in Ob(\mathcal{C})$ then there is a natural bijection $\mathrm{Hom}_C(X, Y) \cong \mathrm{Hom}(h_X, h_Y)$ (natural transformations of functors).

How does this relate back? Because $h_{X\times Y}(Z) = \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) = h_X(Z) \times h_Y(Z)$.

Def. A functor $F : C \to Set$ is *representable* if it is of the form h_X.

So our big result isbasically that $h_X \times h_Y$ is representable.

A lot of abstract mumbling.

2 Next definition of variety

We want to figure out what makes a prevariety a variety. (Note: we are defining variety here, not affine variety or projective variety. They will of course agree, but that's for a little later.)

Consider $\mathcal{X} = \mathbb{A}^1 \coprod_{\mathbb{A}^1 - \{0\}} \mathbb{A}^1$. We want this kind of thing NOT to be a variety.

Def./Prop. A prevariety Y is a *variety* if the following equivalent conditions hold:

1. $\forall$ pairs of morphisms $f, g : X \to Y$ the set $\{x \in X | f(x) = g(x)\} \subset X$ is closed.[1]
2. $\forall$ morphisms $f : X \to Y$, the graph $\Gamma_f : X \xrightarrow{1\times f} X \times Y$ has a closed image.
3. The diagonal $\Delta : Y \xrightarrow{1\times 1} Y \times Y$ has closed image.

Pf. that these are equivalent. (1) $\iff$ (3). To prove $\Rightarrow$, consider $X = Y \times Y$, where f is projection from the first and g is projection from the second part. From (1) we know that $\{(y_1, y_2)|p_1(y_1, y_2) = p_2(y_1, y_2)\}$ is closed, but $p_1(y_1, y_2) = y_1$ and $p_2(y_1, y_2) = y_2$ so this is just the image of Δ.

To prove $\Leftarrow$, Suppose we have $f, g : X \to Y$. Then we have diagram:

$$\begin{array}{ccc} X & \hookleftarrow & (f \times g)^{-1}(\Delta(Y)) \\ \downarrow f \times g & & \downarrow \\ Y \times Y & \hookleftarrow & \Delta(Y) \end{array}$$

[1] This fails in $\mathcal{X}$; let $X = \mathbb{A}^1$ and let f be inclusion via the first set, g via the second. These agree on $\mathbb{A}^1 - \{0\}$ under either inclusion which is not closed.

where $(f \times g)^{-1}(\Delta(Y))$ is just the set we're interested in. We can see this is closed by the diagram.

(1) $\iff$ (2). To prove $\Rightarrow$, let $X' = X \times Y$, let $f' : (x,y) \mapsto f(x)$ and let $g' : (x,y) \mapsto y$. Then the set (1) proves is closed is just $\{(x,y) | y = f(x)\}$ which is what we want to prove.

$2 \Rightarrow 3$ is easy: just use $f = 1$.

Def. A sub-prevariety of X is locally closed subset with induced structure of a prevariety. (Remember, U locally closed means there is some open set V such that $U \subset V$ and U is closed in V.)

Lemma. A sub-prevariety Z of a variety X is a variety.

Pf. Let's check (3). We know $\Delta_X(X)$ is closed in $X \times X$. So we intersect this with $Z \times Z$ and we get $\Delta_Z(Z)$ and this is closed since the map $Z \times Z \hookrightarrow X \times X$ is a morphism.

Lemma. A product of two varieties X and Y is a variety.

Pf. Consider $f \times g, f' \times g' : Z \to X \times Y$. We are interested in the set $\{z \in Z | (f \times g)(z) = (f' \times g')(z)\} = \{z \in Z | f(z) = f'(z)\} \cap \{z \in Z | g(z) = g'(z)\}$ which are each closed sets since X and Y are varieties.

Lemma. An affine variety is a variety.

Pf. Check (3). Let $R = \Gamma(X, O_X)$. We know that $X \times X$ corresponds to $R \otimes_k R$ and Δ corresponds to a map $R \otimes_k R \xrightarrow{\Delta^*} R$ defined by $\Delta^* : (r \otimes s) \mapsto rs$. This is surjective, clearly. Let J be the kernel. Now, $\Delta(X)$ is just $V(J)$, so $\Delta(X)$ is closed.

Lemma. If $f, g : X \to Y$ are two morphisms of prevarieties, then the set $E_{f,g} = \{x \in X | f(x) = g(x)\}$ may not be closed, but is locally closed.

Pf. Say $x \in E$, and let $U \subset Y$ be an affine open set containing $f(y) = g(y)$. Note that $V = f^{-1}(U) \cap g^{-1}(U)$ which is an open set in X containing x. Then $E \cap V = \{v \in V | f(v) = g(v)\}$. But now, $f, g : V \mapsto U$ where U is affine. Thus, $E \cap V$ is closed in V.

To complete this, note that we can give E a finite cover of the form $E \cap V$ where each is closed; thus, E is locally closed (by quasi-compactness).

Prop. Let X be a prevariety. Assume $\forall x, y \in X$ there is an open affine $U \subset X$ containing both x, y. Then X is a variety.

Cor. A projective variety is a variety.

Pf. of Cor. Let $X \subset \mathbb{P}^n$. Say $x, y \in X$. Then there is a line which is not zero at either x or y. Take $\mathbb{P}^n$ minus the zero-locations of this line, and we get an open affine set containing x and y.

Pf. of Prop. Look at $f, g : Z \to X$, consider $E_{f,g}$. We can take the closure $E_{f,g} \subset \overline{E_{f,g}} \subset Z$. Now we choose $z \in \overline{E}$. Let $x = f(z), y = g(z)$. Choose an affine open $U \subset X$

such that $x, y \in U$. Now $z \in f^{-1}(U) \cap g^{-1}(U) \to U$ where U is a variety. We also have $E_{f,g} \cap f^{-1}(U) \cap g^{-1}(U) \subset f^{-1}(U) \cap g^{-1}(U)$ closed, so $z \in E_{f,g}$. Thus, every point in the closure is actually in $E_{f,g}$ so $E_{f,g}$ must be closed.

So now we have consistency of definitions: affine varieties and projective varieties are kinds of varieties.

At this point we're done with all the definitions, and now just want to study the properties of varieties.

3 Dimension

Def. Let X be a variety. The *dimension* of X is denoted $\dim X$ and is the transcendence degree of $k(X)$ over k (denoted $\text{tr.deg}\,_k k(X)$.

Examples.

- $\mathbb{A}^n$ has dimension n.
- $\mathbb{P}^n$ has dimension n because it is covered by $\mathbb{A}^n$s.
- If X has dimension 0, then $X = (\star, k)$, the variety of one element.

Remark. (A topological aside.) Let X be a topological space, and consider chains of irreducible prevarieties. We will see later that the dimension of X is the maximum length of the chain

$$Z_0 \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n = X$$

where $Z_i \subset X$ are irreducible closed prevarieties. The idea here is something like: $\mathbb{A}^3$ has dimension 3 because any closed prevariety in $\mathbb{A}^3$ is a planar manifold, and any closed prevariety on a planar manifold is a linear manifold, and any closed prevariety on a linear manifold is a point.

Prop. Say $Y \subset X$ is a proper closed subvariety of X. Then $\dim Y < \dim X$.

Pf. We can assume X is affine, and let $R = \Gamma(X, O_X)$. Since Y is irreducible, we know that Y corresponds to some prime ideal $P \subset R$ and we can assume $P \neq (0)$ or R since Y is proper.

To restate then, if we have a finitely generated k-algebra R, and $P \subset R$ a nontrivial prime ideal of R, then $\text{tr.deg Frac}\,(R/P) < \text{tr.deg Frac}\,(R) = n$.

Now we prove this. If not, then there exist $x_1, \ldots, x_n \in R$ such that $\overline{x_1}, \ldots, \overline{x_n}$ in R/P are algebraically independent. Choose $0 \neq p \in P$. Then, there is a polynomial $F(Y, X_1, \ldots, X_n)$ such that $F(p, x_1, \ldots, x_n) = 0$. But then when we reduce this polynomial modulo P we get a contradiction since this will give a relation $F'(x_1, \ldots, x_n) = 0$. This completes the proof.