HOMEWORK 8 FOR 18.725, FALL 2015
DUE THURSDAY, NOVEMBER 12 BY 1PM.

(1) (a) Prove that if $X = Spec(A)$ is affine and locally factorial, then $Pic(X)$ is trivial iff A is a UFD.
(b) Let $X \subset \mathbb{P}^n$ be a projective variety. Suppose that the homogeneous coordinate ring of X is a UFD. Show that $Pic(X) \cong \mathbb{Z}$.

(2) (a) Let $X \subset \mathbb{P}^2$ be the plane curve given by $zy^2 = x^3 - x^2 z$. Prove that $Pic^0(X) \cong k^*$.
[Hint: Recall the map $\mathbb{P}^1 \to X$ sending two points (say, 0, ∞) to $x_0 = (0 : 0 : 1)$ and inducing an isomorphism $\mathbb{P}^1 \setminus \{0, \infty\} \to X \setminus x_0$. Pull-back of a degree zero line bundle to $\mathbb{P}^1$ is trivial, while its fibers at 0 and ∞ are identified. The ratio of that identification with the one coming from the trivialization of the line bundle is an element in k^*].
(b) Let $X \subset \mathbb{P}^2$ be the plane curve given by $zy^2 = x^3$. Prove that $Pic^0(X) \cong (k, +)$.
[Hint: Recall the bijective map $\mathbb{P}^1 \to X$ sending, say, 0 to $x_0 = (0 : 0 : 1)$ and inducing an isomorphism $\mathbb{P}^1 \setminus 0 \to X \setminus x_0$. Pull-back of a degree zero line bundle to $\mathbb{P}^1$ is a sheaf L, s.t. on the one hand $L \cong \mathcal{O}$, while on the other hand we have an isomorphism $L \otimes (\mathcal{O}_{\mathbb{P}^1}/(\mathcal{O}_{\mathbb{P}^1}(-2(0))) \cong \mathcal{O}_{\mathbb{P}^1}/(\mathcal{O}_{\mathbb{P}^1}(-2(0))$. Compare the last isomorphism with one coming from the trivialization of L to get an element in k].
(c) In both cases (a,b) describe the kernel of the map $Div_C(X) \to Div_W(X)$.

(3) Let $X = (\mathbb{A}^n \setminus \{0\})/\{\pm 1\}$ $(n > 1)$. Compute $Pic(X)$.
[Hint: the answer is $\mathbb{Z}/2\mathbb{Z}$. Divisors in X are in bijection with divisors in $\mathbb{A}^n$ invariant under the map $x \mapsto -x$. Such a divisor D is the divisor of a function f which is either even or odd; the corresponding divisor on X is principal iff f is even.]

(4) Show that the number of singular points of an irreducible plane curve of degree n can not exceed $\frac{(n-1)(n-2)}{2}$.
[Hint: Use linear algebra to find a degree n curve passing through $\frac{(n-1)(n-2)}{2} + 1$ singular points and as many nonsingular points as possible, then apply Bezout Theorem. Make sure to use that X is irreducible: otherwise the statement fails already for $n = 2$.]

(5) (Optional problem)
(a) Let A be an associative algebra. For $a \in A$ define $ad(a) \in End(A)$ by $ad(a) : x \mapsto ax - xa$. Show that if A is an algebra over $\mathbb{F}_p$ then $ad(a)^p = ad(a^p)$.
(b) Let ∂ be a derivation of an associative $\mathbb{F}_p$-algebra C. Show that ∂^p is also a derivation of C.
[Hint: Apply part (a) to $A = End_{\mathbb{F}_p}(C)$, $a = \partial$ and x the operator of left multiplication by an element in C].

Thus for an affine algebraic variety $X = Spec(C)$ over a field of characteristic $p > 0$ and a vector field $\xi \in Vect(X)$ we get another vector field $\xi^{[p]}$ on X, $\xi^{[p]} \cdot f = \xi \cdot \dots \cdot \xi \cdot f$, where ξ appears p times in the right hand side; $\xi^{[p]}$ is called the restricted power of ξ. The definition clearly extends to nonaffine varieties.

(c) Recall that an irreducible normal curve X is an elliptic curve if the sheaf of Kahler differentials on X is trivial[1] (isomorphic to $\mathcal{O}$). Thus an elliptic curve carries a unique (up to scaling) nonzero vector field ξ. The elliptic curve is called *supersingular* if $\xi^{[p]} = 0$; otherwise it is called ordinary.[2]

Let f be a cubic polynomial with no multiple root. Check that the projective closure of the curve $y^2 = f(x)$ is a supersingular elliptic curve iff x^{p-1} enters $f(x)^{(p-1)/2}$ with zero coefficient ($p \neq 2$).

[1]Oftentimes by an elliptic curve one understands a curve with this property together with a fixed point $x_0 \in X$.

[2]There are several other equivalent forms of the definition. For example, an elliptic curve X over $\mathbb{F}_q$ is supersingular iff $|X(\mathbb{F}_{q^n})|$ is prime to p for all n.

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