18.727 Topics in Algebraic Geometry: Algebraic Surfaces
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ALGEBRAIC SURFACES, LECTURE 5

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1. Examples

(1) If $S \subset \mathbb{P}^n, p \in S$, then projection from p gives a rational map $S \dashrightarrow \mathbb{P}^{n-1}$ defined away from p extending to $\mathrm{Bl}_p S = \tilde{S} \to \mathbb{P}^{n-1}$. For instance, if Q is a smooth quadric in $\mathbb{P}^2$, we get a birational map $Q \dashrightarrow \mathbb{P}^2$ with $|tildeQ \to \mathbb{P}^2$ a morphism. It contracts the two lines passing through p, so $Q = \mathbb{P}^2(2-1)$.

(2) A birational map of $\mathbb{P}^2$ to itself is called a *plane Cremona transformation* e.g. quadratic transformation. One example is $\phi : \mathbb{P} \dashrightarrow \mathbb{P}^2$ given by $(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z})$ It is clearly birational and its own inverse. Let $p = (1 : 0 : 0), q = (0 : 1 : 0), r = (0 : 0 : 1)$. These are the 3 base points of ϕ, and ϕ blows up these points and then blows down the three lines joining them. Similarly, we could take a linear system of 3 independent conics passing through three point p, q, r (non-collinear). Generally, 2 conics passing through p, q, r would have a unique 4th point of intersection, gives the birational map.

(3) Linear systems of cubics: let $p_1, \dots, p_r$ be r distinct points in the plane $(r \leq 6)$ in general position, i.e. no 3 of them are collinear and no six lie on a conic. Let $\pi_r : P_r \to \mathbb{P}^2$ be the blowup of $p_1, \dots, p_r$. Let $d = q - r$. The linear system of cubics passing through $p_1, \dots, p_r$ defines an embedding $j : P_r \to \mathbb{P}^d$, and $S_d = j(P_r)$ is a surface of degree d in $\mathbb{P}^d$, called a del Pezzo surface of degree d. e.g. S_1 is a

Note. Contracting other curves and singularities: let $f : Y \to X$ be a resolution of a normal surface singularity $p \in X$ (i.e. X is normal at p). Then $p \subset X$ is called a *rational singularity*— if $R^1 f_* \mathcal{O}_Y = \mathcal{O}$ and $Y \to X$ is an isomorphism away from $Y \smallsetminus \{f^{-1}(p)\} \to X \smallsetminus \{p\}$, e.g. can include nonsingular p as a rational singularity.

Example. The *duVal singularities* are examples of rational singularities.

A_n $x^2 + y^2 + z^{n+1} = 0$
D_n $x^2 + y^2 z + z^{n-1} = 0.$
E_6 $x^2 + y^3 + z^4 = 0.$

E_6 $x^2 + y^3 + yz^3 = 0$.
E_8 $x^2 + y^3 + z^5 = 0$.

If you resolve these, you get the corresponding Dynkin diagrams for the dual graph of the exceptional curves.

Theorem 1 (Artin Contraction). *A connected set of curves $\{C_i\}$ on a surface Y is the exceptional locus of a rational singularity $p \in X$ iff (a) the intersection matrix (C_i, C_j) is negative definite, and (b) $p_a(D) \leq 0$ for every D supported on $\bigcup C_i$. Note that $p_a(D) = 1 - \chi(\mathcal{O}_D)$ by definition, $2p_a - 2 = D \cdot (D + K)$.*

2. Ruled Surfaces

Definition 1. *A surface X is* ruled *if it birational to $\mathbb{P}^1 \times B$ for a singular projective curve B.*

Let X be a surface, B a nonsingular projective curve.

Definition 2. *A* pencil of curves with base B *on X is a dominant rational map $\pi : X \dashrightarrow B$ s.t. $k(B)$ is alg. closed in $k(X)$.*

Note that this map π is defined on the complement of a finite number of points $x_1, \ldots, x_n$. If π is not regular at these points, they are called base points of the pencil, and the fibers $\{\pi^{-1}(y) | y \in B\}$ is the family of curves of the pencil π. For η the generic point of B, $\pi^{-1}(\eta)$ is called the generic curve of the pencil π.

Definition 3. *A smooth morphism $X \to B$ is called a* geometrically ruled surface *over B if the fibers are all isomorphic to $\mathbb{P}^1$.*

Theorem 2 (Noether-Tsen). *Let $\pi : X \dashrightarrow B$ be a pencil of curves s.t. the generic curve has arithmetic genus zero. Then X is birational to $\mathbb{P}^1 \times B$ (and the generic fiber of π is $\cong \mathbb{P}^1_{k(B)}$). In particular, X is a ruled surface.*

Definition 4. *Let K be a field. K has* property C_r *($r \geq 0$) if for every homogeneous polynomial of degree $d \geq 1$ in $n \geq 2$ variables, it has a nonzero solution in K^n whenever $d^r < n$.*

Remark. Note that K has property C_0 iff K is alg. closed, and finite fields have property C_1. Moreover, if K has property C_r, then K has property C_s for $s \geq r$.

Lemma 1. *If K has property C_1, so does every alg. extension of L of K.*

Proof. We can assume that L/K is finite. Let $F(x)$ be a homogeneous polynomial of degree d in n variables ($d < n$) coefficients in L. Let $f(x) = \mathrm{Norm}_{L/K} F(x)$. By choosing a basis $e_1, \ldots, e_m$ ($m = [L : K]$) of L/K, and setting $x = x_1 e_1 + \cdots + x_m e_m$ we see that f can be expressed as a homogeneous polynomial of degree md in mn variables. Since $d < n, md < m$, we have a nontrivial solution $N_{L/K}(F(x)) = 0 \implies F(x) = 0$. ∎

Proposition 1. *Let k be algebraically closed. Then $k(T)$ (purely transcendental extension in one variable) has property C_1.*

Proof. Let $f(X_1, \ldots, X_n)$ be a homogeneous polynomial of degree $d < n$ in $X_1, \ldots, X_n$ with coefficients in $k(T)$. We may as well assume that the coefficients are in $k[T]$. We'll show $\exists$ a nontrivial solution in $k[T]$. Let $f(x_1, \ldots, X_k) = \sum c_{i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$ for $c_{i_1 \cdots i_n} \in k[T]$. Let $\mu = \max \deg c_{i_1 \cdots i_n}$ over all coefficients of f. Write

$$f = f_0(X_1, \ldots, X_n) + Y f_1(X_1, \ldots, X_n) + \cdots + T^\mu f_\mu(X_1, \ldots, X_n) \tag{1}$$

where $f_i \in k[X_1, \ldots, X_n]$. For new variables $Y_{10}, \ldots, Y_{ns}$ (s to be chosen later), write

$$X_i = Y_{i0} + Y_{i1} T + \cdots + Y_{is} T^s \tag{2}$$

and let

$$\phi(Y_{10}, \cdots, Y_{ns}) = f(\sum_{j=0}^{s} Y_{1j} T^j, \sum Y_{2j} T^j, \ldots, \sum Y_{nj} T^j) \tag{3}$$

This has degree $sd + \mu$ in T. Write it as

$$\phi = \phi_0(Y_{10}, \cdots, Y_{ns}) + T\phi_1(Y_{10}, \cdots, Y_{ns}) + \cdots + T^{sd+\mu} \phi_{sd+\mu}(Y_{10}, \cdots, Y_{ns}) \tag{4}$$

i.e. have $ds+\mu+1$ homogeneous polynomials ϕ_j of degree d in $Y_{10}, \cdots, Y_{ns}$. Since $n > d$, for large enough s, $n(s+1) > ds + \mu + 1$ and there are more variables than equations. Because k is alg. closed, we have a solution in k. □

Proposition 2. *Let k be a field, $\overline{k}$ its alg. closure. Let X be an algebraic curve, proper over k.*

Proof. Riemann-Roch on K_X, straightforward. □

Lemma 2. *If, in addition to the hypothesis of proposition, X also has a k-rational point, then X is k-isomorphic to $\mathbb{P}^1_k$.*

Corollary 1. *Let X have property C_1, and let X be geometrically integral, proper curve over k of arithmetic genus 0. Then $X \cong_k \mathbb{P}^1$.*

of Noether-Tsen. Let η be the generic point of B. By the above, the field $k(\eta) = k(B)$ has property C_1. By assumption, $X_\eta = \pi^{-1}(\eta)$ has arithmetic genus 0. Blowing up X enough times, we get $\phi : X' \to X$ and a morphism $X' \to B$ completing $\pi \circ \phi$. Note that this does not change the generic fiber. By assumption, $k(B)$ is algebraically closed in $k(X)$. We see $X_\eta = (\eta \circ \phi)(\eta)$ is geometrically integral, and therefore is $k(\eta)$-isomorphic to $\mathbb{P}^1_{k(\eta)}$. So $k(X_\eta) \cong k(\eta)(t)$ for t an independent variable over $k(\eta)$, and X is birational to $\mathbb{P}^1 \times B$. □

Theorem 3. *Let $\pi : X \to B$ be a surjective morphism from a surface X to a nonsingular, projective curve B s.t. for some closed point $b \in b$, $\pi^{-1}(b) \cong \mathbb{P}^1$. Then $\exists$ a section $\sigma : B \to X$, an open subset $W \subset B, b \in W$, and an isomorphism $f : \pi^{-1}(W) \to \mathbb{P}^1 \times W$ s.t. the following diagram commutes*

$$\begin{tikzcd} \pi^{-1}(W) \arrow[r, "f"] \arrow[dr, "\pi"'] & \mathbb{P}^1 \times W \arrow[d, "\mathrm{pr}_2"] \\ & W \end{tikzcd} \tag{5}$$

Proof. B is a nonsingular curve, and $\pi_*(\mathcal{O}_X)$ is a torsion-free coherent $\mathcal{O}_B$-module, locally free of finite rank (π is flat and $H^1(\pi^{-1}(b), \mathcal{O}_{\pi^{-1}(b)}) = 0$). By the base change theorem, we see that $H^1(\pi^{-1}(b'), \mathcal{O}_{\pi^{-1}(b')}) = 0$ for b' in a neighborhood V of B, and $\pi_*\mathcal{O}_X \otimes k(b) \to H^0(\pi^{-1}(b'), \mathcal{O}_{\pi^{-1}(b')})$ is an isomorphism for $b' \in V$.

$$\pi^{-1}(b) \cong \mathbb{P}^1 \implies \dim H^0(\pi^{-1}(b'), \mathcal{O}_{\pi^{-1}(b')}) = 1 \tag{6}$$

so $\pi_*\mathcal{O}_X$ is locally free of rank 1, i.e. is $\mathcal{O}_B$. Thus, $k(B)$ is alg. closed in $k(X)$, and $\exists U \subset V$ containing b s.t. $F_{b'} = \pi^{-1}(b')$ is geometrically integral for $b' \in U$. $F_b \cong \mathbb{P}^1$, and the arithmetic genus of $F_{b'}$ does not depend on b', so the generic fiber has arithmetic genus 0 and the closed fibers are $\mathbb{P}^1$. Thus, $F_\eta \cong \mathbb{P}^1_{k(\eta)}$.

This implies that F_η has a rational point over $k(\eta) = k(B)$, and $\exists$ a morphism $\operatorname{Spec} k(B) \to F_\eta$ and therefore $\operatorname{Spec} \mathcal{O}_{B,\eta} \to X$ a B-morphism, giving us a rational section $\sigma : B \dashrightarrow X$. B is a nonsingular curve and X is projective, so σ extends to a morphism. $\sigma : B \to X$ is a section ($\pi \circ \sigma = \mathrm{id}_B$). Let $D = \sigma(B)$. Then $D \cdot F_{b'} = 1$ for $b' \in B$. Let $X' = \pi^{-1}(U)$. Since the fibers of π' are $\mathbb{P}^1$, and $\mathcal{O}_{X'}(D) \otimes \mathcal{O}_{F_{b'}} \cong \mathcal{O}_{F_{b'}(1)}$, we have $\dim_{k(b')} H^0(\mathcal{O}_X(D) \otimes k(b')) = 2$ for $b' \in U$. Again applying the base change theorem, we have $E = \pi_*(\mathcal{O}_{X'}(D))$ a locally free $\mathcal{O}_+U$-module of rank 2 and the canonical homomorphism

$$\pi_*\mathcal{O}_{X'}(b) \otimes k(b') \to H^0(\mathcal{O}_X(D) \otimes \mathcal{O}_{F_{b'}}) \tag{7}$$

is an isomorphism for $b' \in U$. Thus $\pi^*\pi_*\mathcal{O}_{X'}(D) = \pi^*(E) \to \mathcal{O}_{X'}(D)$ is surjective. By the universal property of $\mathbb{P}(E)$, we have a unique U-morphism $u : X' \to \mathbb{P}(E)$ s.t. $u^*(\mathcal{O}_{\mathbb{P}(E)}(D)) \cong \mathcal{O}_{X'}(D)$. It is clear that u is an isomorphism since it is an isomorphism fiber by fiber ($u_b : F_{b'} \xrightarrow{\sim} \mathbb{P}^1(k(b'))$) and take $b \in W \subset U$ small enough to trivialize $\mathbb{P}(E)$. $\square$