Lecture 14
Mobius Inversion Formula, Zeta Functions

Recall:

Mobius function $\mu(n)$ and other functions

$$
\begin{aligned}
\mu(n) &= \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is squarefree} \\ 0 & \text{if } n \text{ is not squarefree} \end{cases} \\
\omega(n) &= \text{number of primes dividing } n \\
U(n) &= 1 \ \forall\, n \\
\mathbb{1}(n) &= \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases} \\
\mu * U &= U * \mu = \mathbb{1} \\
\sum_{d|n} \mu(d) &= \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}
\end{aligned}
$$

Theorem 54. *Let f be an arithmetic function, and $F = f * U$, so that $F(n) = \sum_{d|n} f(d)$. Then $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d}) = \sum_{d|n} \mu(\frac{n}{d}) F(d)$.*

Proof.

$$
\begin{aligned}
F &= f * U \\
F * \mu &= (f * U) * \mu \\
&= f * (U * \mu) \\
&= f * \mathbb{1} = f
\end{aligned}
$$

■

Theorem 55. *If f and F are arithmetic functions and $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$ for all n, then $F(n) = \sum_{d|n} f(d)$ for all n*

Proof.

$$
\begin{aligned}
f &= \mu * F \\
&= F * \mu \\
f * U &= F
\end{aligned}
$$

■

Showed last time that $\phi * U = r_1$ (Recall that $r_1(n) = n^1$ and $\sum_{d|n} \phi(d) = n$), and so Mobius inversion says that

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d} = n\sum_{d|n} \frac{\mu(d)}{d}$$

$$\begin{aligned}
n &= p_1^{e_1} p_2^{e_2} \dots p_r^{e_r} \\
&= n\left(1 - \frac{1}{p_1} \dots - \frac{1}{p_r} + \frac{1}{p_1p_2} + \frac{1}{p_1p_3} \dots + \frac{1}{p_{r-1}p_r} - \frac{1}{p_1p_2p_3} \dots + \frac{(-1)^r}{p_1 \dots p_r}\right) \\
&= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_r}\right) \\
&= n\prod_{r|n}\left(1 - \frac{1}{p}\right)
\end{aligned}$$

Eg. There are 100 consecutive doors in a castle that are all closed. Person 1 opens all doors, person n changes state of every nth door, starting with door number n. At the end, which doors will be open? Door number n changes state $d(n)$ number of times at the end, door n open if $d(n)$ is odd.

$$\begin{aligned}
n &= p_1^{e_1} p_2^{e_2} \dots p_r^{e_r} \\
d(n) &= (e_1 + 1)(e_2 + 1)\dots(e_r + 1)
\end{aligned}$$

$d(n)$ odd if all e are even $\Rightarrow$ at the end, open doors are $1, 4, 9, 16 \dots$

Eg. Describe the arithmetic function $\mu * \mu$ - ie., given $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, what is $(\mu * \mu)(n)$.

Since $\mu * \mu$ is multiplicative, it is enough to know it for a prime power p^e.

$$\begin{aligned}
(\mu * \mu)(n) &= \sum_{d|p^e} \mu(d)\mu\left(\frac{p^e}{d}\right) \\
&= \sum_{0 \le i \le e} \mu(p^i)\mu(p^{e-i}) \\
&= \mu(p^e) + \mu(p)\mu(p^{e-1}) + \mu(p^2)\mu(p^{e-2}) + \dots + \mu(p^{e-1})\mu(p) + \mu(p^e)
\end{aligned}$$

Claim that if $e \ge 3$, then $(\mu * \mu)(p^e) = 0$ because for $0 \le i \le e$, since $i + (e - i) = e \ge 3$, one is ≥ 2 so $\mu(p^i)$ or $\mu(p^{e-i})$ is 0, so all terms vanish.

$$
\begin{aligned}
e = 0 \qquad & (\mu * \mu)(1) & &= \mu(1)\mu(1) = 1 \\
e = 1 \qquad & (\mu * \mu)(p) & &= \mu(p) + \mu(p) = -2 \\
e = 2 \qquad & (\mu * \mu)(p^2) & &= \mu(p^2) + \mu(p)\mu(p) + \mu(p^2) = 1
\end{aligned}
$$

$$
(\mu * \mu)(n) = \begin{cases} 0 & \sum e_i \geq 3 \\ \prod_{i=1}^{r} \begin{cases} -2 & e_1 = 1 \\ 1 & e_1 = 2 \end{cases} & \text{otherwise} \end{cases}
$$

(0 unless n is cube-free)

Conjecture 56 (Mertens Conjecture)**.** *Consider the function $M(n) = \sum_{1 \leq k \leq n} \mu(k)$ - how fast does this grow?*

Mertens and Stieltjes independently conjectured that $|M(n)| < \sqrt{n}$, which would imply the Riemann Hypothesis. The claim that

$$
\text{for every } \varepsilon > 0, \quad \frac{M(n)}{n^{\frac{1}{2}+\varepsilon}} \Rightarrow 0 \text{ as } n \Rightarrow \infty
$$

is equivalent to Riemann Hypothesis.

The strong form was disproved in 1985.

The idea is that $M(n)$ is a sum of ± 1 or 0 terms, expect massive cancelation. If we looked at $\sum \mu(k)$ of only squarefree k only divisible by primes $\leq n$, this sum would be

$$
(1 + \mu(2))(1 + \mu(3)) \ldots (1 + \mu(\phi_k)) = (1-1)(1-1) \cdots = 0
$$

However, this sum includes a lot more integers than just $1 \ldots n$

Zeta Functions - analogue of a generating function.

Suppose we have sequence $a_0, a_1 \ldots$ indexed by positive integers. We introduce **generating function**

$$
A(x) = a_0 + a_1 x + a_2 x^2 \cdots = \sum_{n=0}^{\infty} a_n x^n
$$

Can think of $x \in \mathbb{R}$ or $\mathbb{C}$ close to 0, if a_n's don't grow too fast, will converge to a function of x in some small region around 0.

Eg. $a_0, a_1 \cdots = 1$ then

$$A(x) = 1 + x + x^2 \cdots = \frac{1}{1-x}$$

Eg. $a_n = 2^n + 1 = 2, 3, 5, 9, 17 \ldots$, then

$$\begin{aligned} \sum a_n x^n &= \sum (1 + 2^n) x^n \\ &= \sum x^n + \sum (2x)^n \\ &= \frac{1}{1-x} + \frac{1}{1-2x} \end{aligned}$$

We're interested in multiplicative structure, we'll use a different kind of generating function. Let $f(n)$ be an arithmetic function, s a formal variable - eg, $s \in \mathbb{C}$. Define

$$Z(f, s) = \sum_{n \geq 1} \frac{f(n)}{n^s}$$

It does make sense to talk about when it converges, note that if $f(n)$ doesn't grow too fast, say $|f(n)| \leq n^m$ for some constant m, then sum converges for $\Re(s) > m + 1$ (because $|\frac{f(n)}{n^s}| < n^{-(1+\varepsilon)}$)

Eg. One of the simplest arithmetic functions is $U(n) = 1 \,\forall\, n$. Then

$$Z(U, s) = \sum_{n \geq 1} \frac{1}{n^s} = \zeta(s)$$

which is called the **Riemann Zeta Function**. Tightly connected with distribution of prime numbers. $\zeta(s)$ has an Euler Product. First,

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{n^2} &= \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} \ldots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} \ldots\right)\left(1 + \frac{1}{5^s} \ldots\right) \ldots \\ &= \frac{1}{1 - \frac{1}{2^s}} \frac{1}{1 - \frac{1}{3^s}} \frac{1}{1 - \frac{1}{5^s}} \ldots \\ \zeta(s) &= \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \end{aligned}$$

This factorization is called an Euler Product

Eg. $\mathbb{1}$ is even simpler

$$Z(\mathbb{1}, s) = \sum_{n \geq 1} \frac{\mathbb{1}(n)}{n^s} = 1$$

Eg.

$$\begin{aligned}
\zeta(s)^{-1} &= \prod_{p\text{prime}} (1-p^{-s}) \\
&= \sum_{n\geq 1} \frac{\mu(n)}{n^s} \\
&= Z(\mu, s) \\
\text{Note } Z(f*g, s) &= Z(f,s)Z(g,s) \\
\Rightarrow 1 &= Z(\mathbb{1}, s) \\
&= \underbrace{Z(U,s)}_{\zeta(s)} Z(\mu, s)
\end{aligned}$$

In general, if $f(n)$ is multiplicative, then

$$\begin{aligned}
Z(f,s) &= \sum_{n\geq 1} \frac{f(n)}{n^s} \\
&= \prod_{p\text{ prime}} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \frac{f(p^3)}{p^{3s}} \dots\right)
\end{aligned}$$

and if f is completely multiplicative then $f(p^e) = f(p)^e$, then

$$\begin{aligned}
Z(f,s) &= \prod_{p\text{ prime}} \frac{1}{1-\frac{f(p)}{p^s}} \\
&= \prod_{p\text{ prime}} (1-f(p)p^{-s})^{-1}
\end{aligned}$$

Some neat facts:

1.

$$\zeta(2) = \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

"Proof". Use

$$\begin{aligned}
\frac{\sin x}{x} &= \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots \\
\frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\dots
\end{aligned}$$

so
$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}\dots=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\dots$$
Look at coefficient of x^2:
$$-\frac{1}{6}=-\frac{1}{\pi^2}-\frac{1}{4\pi^2}-\frac{1}{9\pi^2}\dots$$
$$\frac{\pi^2}{6}=\sum\frac{1}{n^2}$$
■

2. Look at coefficient of x^4 to get
$$\frac{1}{120}=\frac{1}{\pi^4}\sum_{0<i<j}\frac{1}{i^2j^2}$$
$$\sum_{0<i<j}\frac{1}{i^2j^2}=\frac{\pi^4}{120}$$
Adding $\zeta(4)=\sum\frac{1}{i^4}$ we get
$$\sum\frac{1}{i^4}+2\sum\frac{1}{i^2j^2}=\zeta(4)+\frac{\pi^4}{60}$$
$$\sum\frac{1}{i^4}+2\sum\frac{1}{i^2j^2}=\left(\sum\frac{1}{i^2}\right)^2=\zeta(2)^2$$
$$\zeta(4)+\frac{\pi^4}{60}=\left(\frac{\pi^2}{6}\right)^2=\frac{\pi^4}{36}$$
$$\zeta(4)=\frac{\pi^4}{90}$$
3. Probability of a random number being squarefree:
$$\frac{\text{number of squarefree integers}\leq x}{x}\to\frac{6}{\pi^2}\text{ as }x\to\infty$$

"Proof".
$$\zeta(2)=\frac{\pi^2}{6}$$
$$=\prod_p\frac{1}{1-\frac{1}{p^2}}$$
$$\Rightarrow\prod_p\left(1-\frac{1}{p^2}\right)=\frac{6}{\pi^2}$$
Probability that random number divisible by $p^2\approx\frac{1}{p^2}$, probability not $\approx 1-\frac{1}{p^2}$. With "independence" $\Rightarrow\prod_p(1-\frac{1}{p^2})=\frac{6}{\pi^2}$ ■

4. Probability that 2 random integers are coprime is $\frac{6}{\pi^2}$

5. Probability that 4 integers a, b, c, d satisfy $(a, b) = (c, d)$ is 40%