18.786 PROBLEM SET 2

(1) Recall that for F a number field (i.e., a finite extension of $\mathbb{Q}$), we have $\mathbb{A}_F$ the topological ring of *adèles*, and its units $\mathbb{A}_F^\times$ form the topological group of *idèles*.

  (a) Show that the canonical map $\widehat{\mathbb{Z}}^\times \times \mathbb{Q}^\times \times \mathbb{R}^{>0} \to \mathbb{A}_{\mathbb{Q}}^\times$ is an isomorphism.

  (b) For F a number field, show that:

$$\Big(\prod_{v \text{ a place of } F} \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times / F^\times$$

  is canonically isomorphic to the class group of F, where if v is an infinite place, $\mathcal{O}_{F_v} := F_v$. Here "canonically" means that you should show that such an isomorphism is *uniquely* characterized by the property that that for each prime ideal $\mathfrak{p}$, the composite map:

$$\mathbb{Z} = \mathcal{O}_{F_\mathfrak{p}}^\times \backslash F_\mathfrak{p}^\times \hookrightarrow \Big(\prod_v \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times \twoheadrightarrow \Big(\prod_v \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times / F^\times \simeq \mathrm{Cl}(F)$$

  maps 1 to the ideal class of $\mathfrak{p}$.

  (c) Similarly, show that the profinite completion of:

$$\Big(\prod_{v \text{ a \textit{finite} place of } F} \mathcal{O}_{F_v}^\times\Big)\backslash \mathbb{A}_F^\times / F^\times$$

  is isomorphic to the *narrow*[1] class group of F.

  (d) For every number field F, show that the canonical map $\big(\widehat{\mathbb{Z}} \times \mathbb{R}\big) \underset{\mathbb{Z}}{\otimes} F \to \mathbb{A}_F$ is an isomorphism.

(2) Recall that $\mathbb{Q}_2^\times/(\mathbb{Q}_2^\times)^2$ has order 8, so $\mathbb{Q}_2$ has 7 (isomorphism classes of) quadratic extensions, corresponding to $\mathbb{Q}_2[\sqrt{d}]$ for d running over a class of coset representatives for the non-squares in $\mathbb{Q}_2^\times$.

  (a) Using the fact that an element of $\mathbb{Z}_2^\times$ is a square if and only if it is congruent to 1 modulo $8\mathbb{Z}_2$, show that these coset representatives can be taken to be $d = 2, 3, 5, 6, 7, 10, 14$.

  (b) By the general structure theory of nonarchimedean local fields, $\mathbb{Q}_2$ admits a single *unramified* quadratic extension. Which value of d above does it correspond to? How does this relate to the explicit formula you found last week for the Hilbert symbol for $\mathbb{Q}_2$?

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[1] This is the group of fractional ideals of F modulo principal ideals defined by *totally positive* elements of $F^\times$, i.e., elements x of $F^\times$ such that for every embedding $F \hookrightarrow \mathbb{R}$, $i(x) > 0$.

(c) For each d as above, find a uniformizer in the field $\mathbb{Q}_2[\sqrt{d}]$, and compute its norm in $\mathbb{Q}_2$.

(3) Recall the definition of the *quaternion algebra* $H_{a,b}$ associated to $a, b \in K$: it is the K-algebra with generators i and j with relations $i^2 = a$, $j^2 = b$ and $ij = -ji$.

(a) Let $K = \mathbb{Q}_2$. Show that every $d \in \mathbb{Q}_2$ admits a square root in $H_{-1,-1}$, i.e., for every d there exists $x \in H$ with $x^2 = d$.

(b) Let K be a nonarchimedean local field of odd residue characteristic, and let $a, b \in K^\times$ with Hilbert symbol $(a, b) = -1$. Show that every element of K admits a square root in $H_{a,b}$.

(4) Show that a local field $K \neq \mathbb{C}$ contains only finitely many roots of unity.

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