LECTURE 17

Proof of the Vanishing Theorem

In this lecture, our goal is to show that, for an extension of nonarchimedean local fields L/K with Galois group G, we have

$$\left[(L \otimes_K K^{\mathrm{unr}})^\times\right]^{\mathrm{t}G} \simeq 0.$$

Recall that this implies that $(L^\times)^{\mathrm{t}G} \simeq \mathbb{Z}^{\mathrm{t}G}[-2]$ (which is the main theorem of cohomological LCFT), which in turn implies that $G^{\mathrm{ab}} \simeq K^\times/\mathrm{N}L^\times$. For now, we'll assume that L/K is totally ramified (a reduction from the general case will occur later), which implies $L^{\mathrm{unr}} = L \otimes_K K^{\mathrm{unr}}$. Last time, we proved that it suffices to show that

$$\hat{H}^0(G_\ell, L^{\mathrm{unr},\times}) = 0 = \hat{H}^1(G_\ell, L^{\mathrm{unr},\times})$$

for all ℓ-Sylow subgroups $G_\ell \subseteq G$, where ℓ is a prime. Note that, if we let $K' := L^{G_\ell}$, then L/K' is a G_ℓ-Galois extension. Thus, we may replace K by K' and G with G_ℓ, so that we may simply assume that G is an ℓ-group (that is, $\#G = \ell^n$ for some n). Now, the latter equality above is simply Hilbert's Theorem 90 (or the generalization thereof shown in Problem 3 of Problem Set 7) for the extension $L^{\mathrm{unr}}/K^{\mathrm{unr}}$, so it remains to show the former, that is, that the norm map

$$\mathrm{N}\colon L^{\mathrm{unr},\times} \to K^{\mathrm{unr},\times}$$

is surjective.

We recall the structure theory of ℓ-groups:

Proposition 17.1. *Let G be an ℓ-group. Then there is a chain of normal subgroups*

$$1 \lhd G_0 \lhd \cdots \lhd G_m = G,$$

such that G_{i+1}/G_i is cyclic for all i.

Proof. The main step is to show that $Z(G) \neq 1$ (i.e., the centralizer of G is non-trivial) if $G \neq 1$. Let G act on itself via the adjoint action, that is, $g \cdot x := gxg^{-1}$ for $g, x \in G$. Then the size of every G-orbit is either 1 or divisible by ℓ. Since

$$\sum_{O \in G\text{-orbits}} \#O = \#G = \ell^n > 1,$$

and the G-orbit of 1 has order 1, ℓ must divide the number of G-orbits of size 1, hence $\#Z(G) \neq 0$. Then, choosing a nontrivial element $x \in Z(G)$, we see that $G/\langle x\rangle$ is a normal subgroup of G, and the result follows by induction. □

Thus, by Galois theory, we have a series of corresponding cyclic extensions

$$L = L_m/L_{m-1}/\cdots/L_0 = K.$$

Since it suffices to show that the norm map is surjective on each of these sub-extensions (since a composition of surjective maps is surjective), we may assume that G is cyclic, say of order n. Recall that $\mathrm{N}\colon L^\times \to K^\times$ is *not* surjective, as we showed $\#\hat{H}^0(G, L^\times) = n$. Now, for each m, let $K \subseteq K_m \subseteq K^{\mathrm{unr}}$ denote the degree-m unramified extension of K. The main step is the following:

CLAIM 17.2. *Let $x \in K^\times$. Then x is in the image of $\mathrm{N}\colon L_m^\times \to K_m^\times$.*

PROOF. Observe that $K^\times/\mathrm{N}L^\times = \mathcal{O}_K^\times/\mathrm{N}\mathcal{O}_L^\times$. Indeed, we have the usual short exact sequence

$$0 \to \mathcal{O}_L^\times \to L^\times \xrightarrow{v} \mathbb{Z} \to 0,$$

which yields the exact sequence

$$0 = \hat{H}^{-1}(\mathbb{Z}) \to \hat{H}^0(\mathcal{O}_L^\times) \hookrightarrow \hat{H}^0(L^\times) \to \hat{H}^0(\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z},$$

and the rightmost map is zero since L/K is totally ramified (and therefore $n \mid v(y)$ for all $y \in K^\times$). Thus, we have an isomorphism $\hat{H}^0(L^\times) \simeq \hat{H}^0(\mathcal{O}_L^\times)$, which is precisely our observation.

We have a commutative diagram

$$\begin{array}{ccccccc}
L^\times & \xrightarrow{\mathrm{N}} & K^\times & \longrightarrow & K^\times/\mathrm{N}L^\times & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
L_m^\times & \xrightarrow{\mathrm{N}} & K_m^\times & \longrightarrow & K_m^\times/\mathrm{N}L_m^\times & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{N}_{L_m/L}} & & \downarrow{\scriptstyle \mathrm{N}_{K_m/K}} & & \downarrow & & \\
L^\times & \longrightarrow & K^\times & \xrightarrow{\mathrm{N}} & K^\times/\mathrm{N}L^\times & \longrightarrow & 0.
\end{array}$$

Now, the composition $K^\times/\mathrm{N}L^\times \to K^\times/\mathrm{N}L^\times$ of induced maps is raising to the nth power, hence 0. We'd like to show that the induced map

$$\mathrm{N}_{K_m/K}\colon K_m^\times/\mathrm{N}L_m^\times \to K^\times/\mathrm{N}L^\times$$

is an isomorphism, which implies that the induced map $K^\times/\mathrm{N}L^\times \to K^\times/\mathrm{N}L^\times$ is 0, proving the claim. By Claim 7.8(3), i.e., our earlier analysis of Herbrand quotients, both groups have order n, hence this map is injective if and only if it is surjective. Moreover, it is equivalent to the map

$$\mathrm{N}_{K_m/K}\colon \mathcal{O}_{K_m}^\times/\mathrm{N}\mathcal{O}_{L_m}^\times \to \mathcal{O}_K^\times/\mathrm{N}\mathcal{O}_L^\times$$

by our observation, and since $\mathrm{N}\colon \mathcal{O}_{K_m}^\times \to \mathcal{O}_K^\times$ is surjective by a proof identical to that of Claim 3.4, this map is surjective too, which completes the proof. □

Again, we have a cyclic group G of order n, and all we need to show is that $\mathrm{N}\colon L^{\mathrm{unr},\times} \to K^{\mathrm{unr},\times}$ is surjective. Applying the claim to $K_m^\times$, we see that every element of $K_m^\times$ is the norm of an element of $L_{m+m'}^\times$, and therefore

$$\mathrm{N}\colon \bigcup_m L_m^\times \to \bigcup_m K_m^\times$$

is surjective. It remains to pass to completions. We know that the image of the map $\mathrm{N}\colon L^{\mathrm{unr},\times} \to K^{\mathrm{unr},\times}$ contains $\bigcup_m K_m^\times$, which is dense, so it is enough to show that the image contains an open neighborhood of 1. Clearly

$$\mathrm{N}(L^{\mathrm{unr},\times}) \supseteq \mathrm{N}(K^{\mathrm{unr},\times}) = (K^{\mathrm{unr},\times})^n,$$

and we saw in Problem 1(a) of Problem Set 1 that every element of $1 + \mathfrak{p}_{K^{\mathrm{unr}}}^{2v(n)+1}$ is an nth power in K^{unr}; this is our desired open neighborhood.

Finally, we prove the general case of the vanishing theorem, where our G-extension L/K of nonarchimedean local fields may not be totally ramified. Let $L/L_0/K$ me the (unique) maximal unramified extension of K inside of L, so that L/L_0 is totally unramified. Let $H := \mathrm{Gal}(L/L_0)$, so that L_0/K is Galois with group G/H.

LEMMA 17.3. *Let X be a complex of G-modules, and suppose we have an exact sequence*

$$1 \to H \to G \to G/H \to 1.$$

If

$$X^{\mathrm{t}H} \simeq 0 \simeq \left(X^{\mathrm{h}H}\right)^{\mathrm{t}G/H},$$

then $X^{\mathrm{t}G} \simeq 0$.

Note that it is not true in general that $\left(X^{\mathrm{t}H}\right)^{\mathrm{t}G/H} = X^{\mathrm{t}G}$! For instance, if H is the trivial group, then

$$X^{\mathrm{t}H} = \mathrm{hCoker}(\mathrm{N}\colon X \to X) = 0,$$

where here $\mathrm{N} = \mathrm{id}_X$.

PROOF. By the first condition, $X_{\mathrm{h}H} \xrightarrow{\text{qis}} X^{\mathrm{h}H}$, so by the second condition and Problem 3 of Problem Set 6,

$$X_{\mathrm{h}G} \simeq \left(X_{\mathrm{h}H}\right)_{\mathrm{h}G/H} \simeq \left(X^{\mathrm{h}H}\right)_{\mathrm{h}G/H} \xrightarrow{\text{qis}} \left(X^{\mathrm{h}H}\right)^{\mathrm{h}G/H} \simeq X^{\mathrm{h}G}.$$

It's easy to check that this quasi-isomorphism is given by the norm map (it is given by the composition of two norm maps), which implies that

$$X^{\mathrm{t}G} = \mathrm{hCoker}\big(X_{\mathrm{h}G} \to X^{\mathrm{h}G}\big)$$

is acyclic, as desired. □

Now, we'd like to show that $\left[(L \otimes_K K^{\mathrm{unr}})^\times\right]^{\mathrm{t}G} \simeq 0$. Recall that we have

$$L \otimes_K K^{\mathrm{unr}} = L \otimes_{L_0} L_0 \otimes_K K^{\mathrm{unr}} = L \otimes_{L_0} \prod_{L_0 \hookrightarrow K^{\mathrm{unr}}} K^{\mathrm{unr}} = \prod_{L_0 \hookrightarrow K^{\mathrm{unr}}} L \otimes_{L_0} K^{\mathrm{unr}}$$

canonically (where the second isomorphism is via the map $\alpha \otimes \beta \mapsto (i(\alpha)\beta)_i$, indexed over embeddings $i\colon L_0 \hookrightarrow K^{\mathrm{unr}}$). We have

$$\left[(L \otimes_K K^{\mathrm{unr}})^\times\right]^{\mathrm{t}H} \simeq \prod_{L_0 \hookrightarrow K^{\mathrm{unr}}} \left[(L \otimes_{L_0} K^{\mathrm{unr}})^\times\right]^{\mathrm{t}H} \simeq \prod_{L_0 \hookrightarrow K^{\mathrm{unr}}} \left[(L \otimes_{L_0} L_0^{\mathrm{unr}})^\times\right]^{\mathrm{t}H} \simeq 0$$

by the totally ramified case (as L_0/K is unramified and L/L_0 is totally ramified), which establishes the first condition of the lemma. To show the second condition, note that

$$\prod_{L_0 \hookrightarrow K^{\mathrm{unr}}} \left[(L \otimes_{L_0} K^{\mathrm{unr}})^\times\right]^{\mathrm{h}H} = \prod_{L_0 \hookrightarrow K^{\mathrm{unr}}} K^{\mathrm{unr},\times} \simeq K^{\mathrm{unr},\times}[G/H]$$

as a G/H-module (once we fix an embedding $L_0 \hookrightarrow K^{\mathrm{unr}}$). But as shown in Problem 1(e) of Problem Set 7, Tate cohomology vanishes for induced modules (thus, the equality above is irrelevant, as we just needed a product over such embeddings to construct an induced G/H-module). Lemma 17.3 then yields the desired result.