18.906: Problem Set V

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet.

Extra credit for finding mistakes and telling me about them early!

22. $\varnothing$

23. **(a)** Suppose F_*C_* is a first-quadrant filtered complex such that $E^1_{*,*} = 0$. Show that $H_*(C_*) = 0$ in the following way. Let $c \in H_n(C_*)$. Show that it is represented by a cycle $z \in F_sC_n$ for some s. Then make an appropriate sequence of choices to come up with a class $y \in C_{n+1}$ with $dy = z$.

(b) Show by example that if we omit the either the first or the third condition in the definition of "first quadrant" (Definition 61.2) then the conclusion of **(a)** may fail.

(c) Give a counterexample to Corollary 61.4 if you keep the second and third conditions of Definition 61.2 but omit the first one.

24. **(a)** Suppose that $p : E \downarrow S^n$ is a fibration over the n-sphere, with fiber F. There is a natural long exact sequence involving $H_*(E)$ and $H_*(F)$, analogous to the Gysin sequence. Derive it carefully from the Serre spectral sequence. (The case $n = 1$ requires special attention.)

(b) Let $f : S^2 \to S^2$ be a map of degree 2. Compute the homology of its homotopy fiber.

(c) Let $f : \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be a map inducing multiplication by 2 in H_2, and let $\mathbb{C}P^\infty \xrightarrow{\simeq} E \xrightarrow{p} \mathbb{C}P^\infty$ be a factorization of f into a homotopy equivalence followed by a fibration. Determine the behavior of the homology Serre spectral sequence for p.

25. Let C be a chain complex such that C_n is free for each n (for example, the singular chains on a space or the cellular chains on a CW complex). Let p be a prime number. The *Prüfer group* is

$$\mathbb{Z}_{p^\infty} = \bigcup \mathbb{Z}/p^s\mathbb{Z} = \mathbb{Z}[1/p]/\mathbb{Z}$$

It's filtered by

$$F_s\mathbb{Z}_{p^\infty} = \ker(p^{s+1}|\mathbb{Z}_{p^\infty}) = \mathbb{Z}/p^{s+1}\mathbb{Z}\,.$$

Filter $C \otimes \mathbb{Z}_{p^\infty}$ accordingly.

(a) Show that in the resulting spectral sequence

$$E^1_{s,t} = \begin{cases} H_{s+t}(C \otimes \mathbb{F}_p) & \text{if } s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Also describe the corresponding exact couple.

(b) This is not a first quadrant spectral sequence; which part of the (s,t)-plane does it live in? What if $C_n = 0$ for $n < 0$? Show that nevertheless it converges,

$$E^r_{s,t} \underset{s}{\Longrightarrow} H_{s+t}(C \otimes \mathbb{Z}_{p^\infty})\,.$$

in the sense that one can define $E^\infty_{s,t}$ and these groups form the associated graded groups of a filtration on $H_{s+t}(C \otimes \mathbb{Z}_{p^\infty})$ that is "exhaustive" in the sense that

$$F_{-1} = 0 \quad \text{and} \quad \lim_{s\to\infty} F_s H_*(C \otimes \mathbb{Z}_{p^\infty}) = H_*(C \otimes \mathbb{Z}_{p^\infty})\,.$$

The "abutment" of this spectral sequence – the group it is trying to converge to – is of interest because it determines the p-torsion in $H_*(C)$ by means of the long exact sequence

$$\cdots \to H_{n+1}(C \otimes \mathbb{Z}_{p^\infty}) \to H_n(C) \to H_n(C)[1/p] \to H_n(C \otimes \mathbb{Z}_{p^\infty}) \to \cdots\,.$$

(c) If $H_n(C)$ is finitely generated, show that the structure of the p-torsion in it is completely described by the spectral sequence.

(d) Show that the data contained in this spectral sequence can be captured by a simpler "singly graded spectral sequence," with

$$E^1_n = H_n(C \otimes \mathbb{Z}/p\mathbb{Z})\,, \quad E^{r+1}_n = H_n(E^r_*, \beta^r)\,, \quad \beta^r : E^r_n \to E^r_{n-1}\,.$$

This is the "Bockstein spectral sequence." What is $\beta^1 : H_n(C \otimes \mathbb{Z}/p\mathbb{Z}) \to H_{n-1}(C \otimes \mathbb{Z}/p\mathbb{Z})$?

26. Let $p : E \to B$ be a fibration, A a subspace of B, and $* \in E_A$. Let E_A denote the restriction or pullback of E to A. Show that the projection maps induces an isomorphism

$$\pi_n(E, E_A, *) \to \pi_n(B, A, *)$$

is an isomorphism for $n \geq 1$.

27. Analyze the cohomology Serre spectral sequence for the same fibration that you studied in **24 (c)**: Differentials? Extensions?