Chapter 1

Basic homotopy theory

1 Limits, colimits, and adjunctions

Limits and colimits

I want to begin by developing a little more category theory. I still refer to the classic text *Categories for the Working Mathematician* by Saunders Mac Lane [20] for this material.

Definition 1.1. Suppose $\mathcal{I}$ is a small category (so that it has a *set* of objects), and let $\mathcal{C}$ be another category. Let $X : \mathcal{I} \to \mathcal{C}$ be a functor. A *cone under* X is a natural transformation e from X to a constant functor; to be explicit, this means that for every object i of $\mathcal{I}$ we have a map $e_i : X_i \to Y$, and these maps are compatible in the sense that for every $f : i \to j$ in $\mathcal{I}$ the following diagram commutes:

$$
\begin{array}{ccc}
X_i & \xrightarrow{\ e_i\ } & Y \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle =} \\
X_j & \xrightarrow{\ e_j\ } & Y
\end{array}
$$

A *colimit* of X is an initial cone (L, t_i) under X; to be explicit, this means that for any cone (Y, e_i) under X, there exists a unique map $h : L \to Y$ such that $h \circ t_i = e_i$ for all i.

Any two colimits are isomorphic by a unique isomorphism compatible with the structure maps; but existence is another matter. Also, as always for category theoretic concepts, some examples are in order.

Example 1.2. If $\mathcal{I}$ is a discrete category (that is, the only maps are identity maps; $\mathcal{I}$ is entirely determined by its set of objects), the colimit of a functor $\mathcal{I} \to \mathcal{C}$ is the coproduct in $\mathcal{C}$ (if this coproduct exists!).

Example 1.3. In Lecture 23 we discussed directed posets and the direct limit of a directed system $X : \mathcal{I} \to \mathcal{C}$. The colimit simply generalizes this to arbitrary indexing categories rather than restricting to directed partially ordered sets.

Example 1.4. Let G be a group; we can view this as a category with one object, where the morphisms are the elements of the group and composition is given by the group structure. If $\mathcal{C} = \mathbf{Top}$ is the category of topological spaces, a functor $G \to \mathcal{C}$ is simply a group action on a topological space X. The colimit of this functor is the orbit space of the G-action on X (together with the projection map to the orbit space).

Similarly, a functor from G into vector spaces over a field k is a representation of G on a vector space. Question for you: What is the colimit in this case?

Example 1.5. Let $\mathcal{I}$ be the category whose objects and non-identity morphisms are described by the following directed graph:

$$b \leftarrow a \rightarrow c \,.$$

The colimit of a diagram $\mathcal{I} \rightarrow \mathcal{C}$ is called a *pushout*. With $\mathcal{C} = \mathbf{Top}$, again, a functor $\mathcal{I} \rightarrow \mathcal{C}$ is determined by a diagram of spaces:

$$A \xleftarrow{f} B \xrightarrow{g} C \,.$$

The colimit of such a functor is just the pushout $B \cup_A C := B \sqcup C/\sim$, where $f(a) \sim g(a)$ for all $a \in A$. We have already seen this in action before: a special case of this construction appears in the process of attaching cells to build up a CW-complex.

If $\mathcal{C}$ is the category of groups, instead, the colimit of such a functor is the free product quotiented out by a certain relation; this is called the *amalgamated free product*.

Example 1.6. Suppose $\mathcal{I}$ is the category with two objects and two parallel morphisms:

$$a \rightrightarrows b \,.$$

The colimit of a diagram $\mathcal{I} \rightarrow \mathcal{C}$ is called the *coequalizer* of the diagram. If $\mathcal{C} = \mathbf{Top}$, the coequalizer of $f, g : A \rightrightarrows B$ is the quotient of B by the equivalence relation generated by $f(a) \sim g(a)$ for $a \in A$.

One can also consider cones *over* a diagram $X : \mathcal{I} \rightarrow \mathcal{C}$: this is simply a cone in the opposite category.

Definition 1.7. The *limit* of a diagram $X : \mathcal{I} \rightarrow \mathcal{C}$ is a terminal object in cones over X.

Exercise 1.8. Revisit the examples provided above: what is the limit of each diagram? For instance, a product is a limit over a discrete category, and the limit of a group action is just the fixed points. A diagram indexed by the category $b \rightarrow a \leftarrow c$ is a diagram $B \xrightarrow{f} A \xleftarrow{g} C$, and its limit is the "pullback," denoted $B \times_A C$. In **Set**, or **Top**,

$$B \times_A C = \{(b, c) \in B \times C : f(b) = g(c) \in A\} \,.$$

Definition 1.9. A category $\mathcal{C}$ is *cocomplete* if all functors from small categories to $\mathcal{C}$ have colimits. Similarly, $\mathcal{C}$ is *complete* if all functors from small categories to $\mathcal{C}$ have limits.

All the large categories we typically deal with are both cocomplete and complete; in particular both **Set** and **Top** are, as well as algebraic categories like **Gp** and $R - \mathbf{Mod}$.

Adjoint functors

The notion of a colimit as a special case of the more general concept of an adjoint functor, as long as we are dealing with a cocomplete category.

Let's write $\mathcal{C}^{\mathcal{I}}$ for the category of functors from $\mathcal{I}$ to $\mathcal{C}$, and natural transformations between them. There is a functor $c : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$, given by sending any object to the constant functor taking on that value. The process of taking the colimit of a diagram supplies us with a functor $\mathrm{colim}_{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}$. (To be precise, we pick a specific colimit for each diagram, and then observe that a natural

transformation of diagrams canonically defines a morphism between the corresponding colimits; and that these morphisms compose correctly.) We can characterize this functor via the formula

$$\mathcal{C}(\operatorname*{colim}_{i \in \mathcal{I}} X_i, Y) = \mathcal{C}^{\mathcal{I}}(X, c_Y),$$

where X is any functor from $\mathcal{I}$ to $\mathcal{C}$, Y is any object of $\mathcal{C}$, and c_Y denotes the constant functor with value Y. This formula is reminiscent of the adjunction operation in linear algebra, and is in fact our first example of a category-theoretic adjunction.

Definition 1.10. Let $\mathcal{C}, \mathcal{D}$ be categories, and suppose given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. An *adjunction between F and G* is an isomorphism

$$\mathcal{D}(FX, Y) = \mathcal{C}(X, GY)$$

that is natural in X and Y. In this situation, we say that F is a *left adjoint* of G and G is a *right adjoint* of F.

    This notion was invented by the late MIT Professor Dan Kan, in 1958.

    We've already seen one example of adjoint functors. Here is another one.

Example 1.11 (Free groups)**.** There is a forgetful functor $u : \mathbf{Grp} \to \mathbf{Set}$. Any set X gives rise to a group FX, the free group on X. It is determined by a universal property: For any group Γ, set maps $X \to u\Gamma$ are the same as group homomorphisms $FX \to \Gamma$. This is exactly saying that the free group functor the left adjoint to the forgetful functor u.

    In general, "free objects" come from left adjoints of forgetful functors.

    As a general notational practice, try to write the left adjoint as the top arrow:

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G \qquad \text{or} \qquad G : \mathcal{D} \leftrightarrows \mathcal{C} : F.$$

    These examples suggest that if a functor F has a right adjoint then any two right adjoints are canonically isomorphic. This is true and easily checked. We'll always speak of *the* right adjoint, or *the* left adjoint.

Lemma 1.12. *Suppose that*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{E}$$

is a composable pair of adjoint functors. Then $F'F, GG'$ form an adjoint pair.

Proof. Compute:

$$\mathcal{E}(F'FX, Z) = \mathcal{D}(FX, G'Y) = \mathcal{C}(X, GG'Y). \qquad \square$$

Proposition 1.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F admits a right adjoint then it preserves colimits, in the sense that if $X : \mathcal{I} \to \mathcal{C}$ is a diagram in $\mathcal{C}$ with colimit cone $X \to c_L$, then $F \circ X \to F(c_L)$ is a colimit cone in $\mathcal{D}$. Dually, if F admits a left adjoint then it preserves limits.*

Proof. This follows from the lemma. The adjoint pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ induces an adjoint pair

$$F : \mathcal{C}^{\mathcal{I}} \rightleftarrows \mathcal{D}^{\mathcal{I}} : G.$$

Clearly $c_{GY} = Gc_Y$; this is an equality of right adjoints, so the corresponding left adjoints must be equal:

$$
\begin{array}{ccc}
\mathcal{C}^{\mathcal{I}} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{D}^{\mathcal{I}} \\
c \Big\Updownarrow \mathrm{colim} & & c \Big\Updownarrow \mathrm{colim} \\
\mathcal{C} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathcal{D}
\end{array}
$$

That is to say, $\mathrm{colim}\, FX = F\,\mathrm{colim}\, X$. $\square$

For example, the free group on a disjoint union of sets is the free product of the two groups (which is the coproduct in the category of groups). The dual statement says, for example, that the product (in the category of groups) of groups is a group structure on the product of their underlying sets.

The Yoneda lemma

An important and rather Wittgensteinian principle in category theory is that an object is determined by the collection of all maps out of it. The Yoneda lemma is a way of making this precise. Observe that for any $X \in \mathcal{C}$ the association $Y \mapsto \mathcal{C}(X, Y)$ gives us a functor $\mathcal{C} \to \mathbf{Set}$. This functor is said to be *corepresented* by X. Suppose that $G : \mathcal{C} \to \mathbf{Set}$ is any functor. An element $x \in G(X)$ determines a natural transformation

$$\theta_x : \mathcal{C}(X, -) \to G$$

in the following way. Let $Y \in \mathcal{C}$ and $f : X \to Y$, and define

$$\theta_x(f) = f_*(x) \in G(Y).$$

Lemma 1.14 (Yoneda lemma)**.** *The association $x \mapsto \theta_x$ provides a bijection*

$$G(X) \overset{\cong}{\to} \mathrm{nt}(\mathcal{C}(X, -), G).$$

Proof. The inverse sends a natural transformation $\theta : \mathcal{C}(X, -) \to G$ to $\theta_X(1_X) \in G(X)$. $\square$

In particular, if G is also corepresentable – $G = \mathcal{C}(Y, -)$, say – then

$$\mathrm{nt}(\mathcal{C}(X, -), \mathcal{C}(Y, -)) \cong \mathcal{C}(Y, X).$$

That is, each natural transformation $\mathcal{C}(X, -) \to \mathcal{C}(Y, -)$ is induced by a unique map $Y \to X$. Consequently any natural isomorphism $\mathcal{C}(X, -) \overset{\cong}{\to} \mathcal{C}(Y, -)$ is induced by a unique isomorphism $Y \overset{\cong}{\to} X$.

2   Cartesian closure and compactly generated spaces

The category of topological spaces has a lot to recommend it, but it does not accommodate constructions from algebraic topology gracefully. For example, the product of two CW complexes may fail to have a CW structure. (This is a classic example due to Clifford Dowker, 1952, nicely explained in [12, Appendix]. The CW complexes involved are one-dimensional!) This is closely related to the observation that if $X \to Y$ is a quotient map, the induced map $W \times X \to W \times Y$ may fail to be a quotient map.

It turns out that these problems can be avoided by working in a carefully designed subcategory of **Top**, the category k**Top** of "compactly generated spaces." The key idea is that the unwanted behavior of **Top** is related to the fact that there isn't a well-behaved topology on the set of continuous maps between two spaces. The compact-open topology is available to us – and we'll recall it later. But it suffers from some defects. To clarify how a mapping object should behave in an ideal world, I want to make another category-theoretical digression. Again, Mac Lane's book [20] is a good reference.

Cartesian closure

How *should* function objects behave? In the category **Set**, for example, the set of maps from X to Y can be characterized by the natural bijection

$$\mathbf{Set}(W \times X, Y) = \mathbf{Set}(W, \mathbf{Set}(X, Y))$$

under which $f : W \times X \to Y$ corresponds to $w \mapsto (x \mapsto f(w, x))$ and $g : W \to \mathbf{Set}(X, Y)$ corresponds to $(w, x) \mapsto g(w)(x)$. This suggests the following definition.

Definition 2.1. Let $\mathcal{C}$ be a category with finite products. It is *Cartesian closed* if for any object X in $\mathcal{C}$, the functor $- \times X$ has a right adjoint.

We'll write the right adjoint to $- \times X$ using exponential notation,

$$Y \mapsto Y^X,$$

so that there is a bijection natural in the pair (W, Y):

$$\mathcal{C}(W \times X, Y) = \mathcal{C}(W, Y^X).$$

In a Cartesian closed category, Y^X serves as a "mapping object" from X to Y. Let me convince you that this is reasonable. Take $Y = W \times X$: the identity map on $W \times X$ then corresponds to a map

$$\eta_W : W \to (W \times X)^X.$$

Take $W = Y^X$: the identity map $Y^X \to Y^X$ corresponds to a map

$$\epsilon_Y : Y^X \times X \to Y.$$

These maps are natural transformations. In the example of **Set**, the first is given by

$$w \mapsto (x \mapsto (w, x)), \quad \text{inclusion of a slice},$$

and the second is given by

$$(f, x) \mapsto f(x), \quad \text{evaluation}.$$

Here are some direct consequences of Cartesian closure. Note: the assumption that finite products exist in $\mathcal{C}$ includes the case in which the indexing set is empty, in which case the universal property of the product characterizes the terminal object of $\mathcal{C}$, which thus exist in a Cartesian closed category. We'll denote it by $*$. You might call $\mathcal{C}(*, X)$ the "set of points" in X.

Proposition 2.2. *Let $\mathcal{C}$ be Cartesian closed.*
(1) $(X, Z) \mapsto Z^X$ extends canonically to a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, and the bijection $\mathcal{C}(X \times Y, Z) = \mathcal{C}(Y, Z^X)$ is natural in all three variables.
(2) $\mathcal{C}(X, Z) = \mathcal{C}(*, Z^X)$.
(3) $X \times -$ preserves colimits: If $Y : \mathcal{I} \to \mathcal{C}$ has a colimit, then the natural map $X \times Y \to X \times \mathrm{colim}\, Y$ is a colimit cone.
(4) $-^X$ preserves limits: if $Z : \mathcal{I} \to \mathcal{C}$ has a limit, then the natural map $(\lim Z)^X \to (Z^X)$ is a limit cone.

Many otherwise well-behaved categories are not Cartesian-closed. A category is *pointed* if it has an initial object $\varnothing$ and a final object $*$, and the unique map $\varnothing \to *$ is an isomorphism. There are many pointed categories! – abelian groups **Ab** and groups **Gp**, for example. By (2), the only way a pointed category can be Cartesian closed is if there is exactly one map between any two objects.

k-spaces

The category **Top** is not Cartesian closed. We can see this using the observation (for which see for example MIT professor emeritus Jim Munkres's *Topology* [30]) that if $X \to Y$ is a quotient map, the induced map $W \times X \to W \times Y$ may fail to be a quotient map. We can characterize quotient maps in **Top** categorically using the following definition.

Definition 2.3. An *effective epimorphism* in a category $\mathcal{C}$ is a map $X \to Y$ in $\mathcal{C}$ such that the pullback $X \times_Y X$ exists and the map $X \to Y$ is the coequalizer of the two projection maps $X \times_Y X \to X$.

Lemma 2.4. *A map in **Top** is a quotient map if and only if it is an effective epimorphism.*

So item 3 of Proposition 2.2 shows that, sadly, **Top** is not Cartesian closed.

On the other hand, Henry Whitehead showed that crossing with a locally compact Hausdorff space *does* preserve quotient maps. This will often suffice, but often not: for example CW complexes may fail to be locally compact. And the convenience of working in a Cartesian closed category is compelling.

Inspired by Whitehead's theorem, we agree to accept only properties of a space that can be observed by mapping compact Hausdorff spaces into it.

Definition 2.5. Let X be a space. A subspace $F \subseteq X$ is said to be *compactly closed*, or k-closed, if for any map $k : K \to X$ from a compact Hausdorff space K the preimage $k^{-1}(F) \subseteq K$ is closed.

It is clear that any closed subset is compactly closed, but there might be compactly closed sets that are not closed in the topology on X. This motivates the definition of a k-space:

Definition 2.6. A topological space X is *compactly generated* or is a *k-space* if every compactly closed set is closed.

The k comes from the German "*kompact*," though it might have referred to the general topologist John Kelley, who explored this condition.

A more categorical characterization of this property is: X is compactly generated if and only if a map $X \to Y$ is continuous precisely when for every compact Hausdorff space K and map $k : K \to X$ the composite $K \to X \to Y$ is continuous. For instance, compact Hausdorff spaces are k-spaces. First countable spaces (so for example metric spaces) and CW-complexes are also k-spaces.

While not all topological spaces are k-spaces, any space can be "k-ified." The procedure is simple: endow the underlying set of a space X with an new topology, one for which the closed sets are precisely the sets that are compactly closed with respect to the original topology. You should check that this is indeed a topology on X. The resulting topological space is denoted kX. This construction immediately implies that the identity $kX \to X$ is continuous, and is the terminal map to X from a k-space.

Let $k\mathbf{Top}$ be the category of k-spaces, as a full subcategory of $\mathbf{Top}$. We will write $j : k\mathbf{Top} \to \mathbf{Top}$ for the inclusion functor. The process of k-ification gives a functor $k : \mathbf{Top} \to k\mathbf{Top}$ with the property that

$$k\mathbf{Top}(X, kY) = \mathbf{Top}(jX, Y).$$

This is another example of an adjunction! In this case the unit $\eta : X \to kjX$ is a homeomorphism.

We can conclude from this that limits in $k\mathbf{Top}$ may be computed by k-ifying limits in $\mathbf{Top}$: For any functor $X : \mathcal{I} \to k\mathbf{Top}$,

$$\lim{}^{k\mathbf{Top}} X \xrightarrow{\cong} \lim{}^{k\mathbf{Top}} kjX \xleftarrow{\cong} k \lim{}^{\mathbf{Top}} jX.$$

The second map is an isomorphism because k is a right adjoint. In particular, the product in $k\mathbf{Top}$ is formed by k-ifying the product in $\mathbf{Top}$. Similarly, colimit (in $k\mathbf{Top}$) of any diagram of k-spaces can be computed by k-ifying the colimit in $\mathbf{Top}$:

$$\operatorname{colim}^{k\mathbf{Top}} X \xrightarrow{\cong} kj \operatorname{colim}^{k\mathbf{Top}} X \xleftarrow{\cong} k \operatorname{colim}^{\mathbf{Top}} jX.$$

The second map is an isomorphism because j is a left adjoint.

The category $k\mathbf{Top}$ has good categorical properties inherited from $\mathbf{Top}$: it is a complete and cocomplete category. In fact it has even better categorical properties than $\mathbf{Top}$ does:

Proposition 2.7. *The category $k\mathbf{Top}$ is Cartesian closed.*

Proof. See [39, 10]. □

I owe you a description of the mapping object Y^X. It consists of the set of continuous maps from X to Y endowed with a certain topology. For general topological spaces X and Y, the set $\mathbf{Top}(X, Y)$ can be given the "compact-open topology": a basis for open sets for the compact-open topology is given by

$$V(F, U) = \{ f : X \to Y : f(F) \subseteq U \}$$

where F runs over compact subsets of X and U runs over open subsets of Y. This space is not generally compactly generated, however, and does not serve as a right adjoint to the product.

If X and Y are k-spaces, it's natural to make a slight modification: To start with, replace the compact subsets F in this definition by "k-compact" subsets, that is, subsets that are compact from the perspective of compact Hausdorff spaces: A subset $F \subseteq X$ is k-compact if there exists a compact Hausdorff space K and a map $k : K \to X$ such that $k(K) = F$. This is to overcome the sad fact that there are compact spaces that do not accept surjections from compact Hausdorff spaces.

The sets $V(F, U)$ where F runs over k-compact subsets of X and U runs over open subsets of Y form the basis of a new topology on $\mathbf{Top}(X, Y)$. Even if we assume that X and Y are k-spaces, this new topology may not be compactly generated. But we know what to do: k-ify it. This defines a k-space Y^X, and this turns out to witness the fact that $k\mathbf{Top}$ is Cartesian closed.

3   Basepoints and the homotopy category

More on k-spaces

The ancients (mainly Felix Hausdorff, in 1914) came up with a good definition of a topology – but k-spaces are better!

Most spaces encountered in real life are k-spaces already, and many operations in **Top** preserve the subcategory k**Top**.

Proposition 3.1 (see [39, 10])**.** (1) *Any locally compact Hausdorff space is compactly generated.*

(2) *Quotient spaces and closed subspaces of compactly generated spaces are compactly generated.*

(3) *If X is a locally compact Hausdorff space and Y is compactly generated then $X \times Y$ is again compactly generated.*

(4) *The colimit of any diagram of compactly generated spaces is compactly generated.*

As a result of (4), in the homeomorphism

$$k \operatorname{colim}^{\mathbf{Top}} j X_\bullet \to \operatorname{colim}^{k\mathbf{Top}} X_\bullet$$

that we considered in the last lecture, the space $\operatorname{colim}^{\mathbf{Top}} j X_\bullet$ is in fact already compactly generated; no k-ification is necessary – the colimit constructed in **Top** is the same as the colimit constructed in k**Top**.

When we say "space" in this course, we will always mean k-space, and the various constructions – products, mapping spaces, and so on – will take place in k**Top**.

I should add that there is a version of the Hausdorff condition that is well suited to the compactly generated setting. Check out the sources [39, 10] for this.

Here's a simple example of how useful the formation of mapping spaces can be. We already know that a *homotopy* between maps $f, g : X \to Y$ is a map $h : I \times X \to Y$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X & & \\
\text{in}_0 \downarrow & \searrow^{f} & \\
I \times X & \xrightarrow{\ h\ } & Y \\
\text{in}_1 \uparrow & \nearrow_{g} & \\
X & &
\end{array}
$$

We write $f \sim g$ to indicate that f and g are homotopic. This is an equivalence relation on the set $\mathbf{Top}(X, Y)$, and we write

$$[X, Y] = \mathbf{Top}(X, Y)/\sim$$

for the set of homotopy classes of maps from X to Y.

The maps f and g are points in the space Y^X, and the homotopy h is the same thing as a path $\hat{h} : I \to Y^X$ from f to g. So

$$[X, Y] = \pi_0(Y^X) \, .$$

Another important feature of k-spaces is this:

Theorem 3.2 (see [12, Theorem A.6])**.** *Let X and Y be CW-complexes with skeleta $\operatorname{Sk}_i X$ and $\operatorname{Sk}_j Y$. Then the k-space product $X \times Y$ admits the structure of a CW complex in which*

$$\operatorname{Sk}_n(X \times Y) = \bigcup_{i+j=n} \operatorname{Sk}_i X \times \operatorname{Sk}_j Y \, .$$

Basepoints

To talk about the fundamental group and higher homotopy groups we have to get basepoints into the picture.

A *pointed space* is a space X together with a specified point in it, to be called the *basepoint*. It is conventionally denoted by $*$, though other symbols may be used as well. The term "basepoint" leads some people refer to "based spaces," but to my ear this makes it sound as if we are doing chemistry, or worse, and I prefer "pointed." We may put restrictions on the choice of basepoint; for example we may require that $\{*\}$ be a closed subset. We will put a further restriction on $\{*\} \hookrightarrow X$ in 6.2.

This gives a category $k\mathbf{Top}_*$ where the morphisms respect the basepoints. This category is complete and cocomplete. For example

$$(X, *) \times (Y, *) = (X \times Y, (*, *))$$

The coproduct is not the disjoint union; which basepoint would you pick? So you identify the two basepoints; the coproduct in $k\mathbf{Top}_*$ is the "wedge"

$$X \vee Y = \frac{X \sqcup Y}{*_X \sim *_Y}.$$

The one-point space $*$ is the terminal object in $k\mathbf{Top}_*$, as in $k\mathbf{Top}$, but it is also *initial* in $k\mathbf{Top}_*$: $k\mathbf{Top}_*$ is a pointed category. As we saw, this precludes it from being Cartesian closed. But we still know what we would like to take as a "mapping object" in $k\mathbf{Top}_*$: Define Y_*^X to be the subspace of Y^X consisting of the pointed maps. In general we may have to k-ify this subspace, but if $\{*\}$ is closed in Y then Y_*^X is closed in Y^X and hence is already a k-space. As a replacement for Cartesian closure, let's ask: For fixed $X \in k\mathbf{Top}_*$, does the functor $Y \mapsto Y_*^X$ have a left adjoint? This would be an analogue in $\mathbf{Top}_*$ of the functor $A \otimes -$ in $\mathbf{Ab}$. Compute:

$$
\begin{array}{ccc}
k\mathbf{Top}(W, Y^X) & = & \{f : W \times X \to Y\} \\
\cup| & & \cup| \\
k\mathbf{Top}(W, Y_*^X) & = & \{f : f(w, *) = * \ \forall\, w \in W\} \\
\cup| & & \cup| \\
k\mathbf{Top}_*(W, Y_*^X) & = & \left\{f : \begin{array}{ll} f(w, *) = * & \forall\, w \in W \\ f(*, x) = * & \forall\, x \in X \end{array}\right\}.
\end{array}
$$

So the map $W \times X \to Y$ corresponding to $f : W \to Y_*^X$ sends the wedge $W \vee X \subseteq X \times W$ to the basepoint of Y, and hence factors (uniquely) through the *smash product*

$$W \wedge X = \frac{W \times X}{W \vee X}$$

obtained by pinching the "axes" in the product to a point. We have an adjoint pair

$$- \wedge X : k\mathbf{Top}_* \rightleftarrows k\mathbf{Top}_* : (-)_*^X.$$

A good way to produce a pointed space is to start with a pair (X, A) (with A a closed subspace of X) and collapse A to a point. Thus

$$k\mathbf{Top}_*(X/A, Y) = \{f : X \to Y : f(A) \subseteq \{*\}\}.$$

What if $A = \varnothing$? Then the condition is empty, so

$$k\mathbf{Top}_*(X/\varnothing, Y) = \mathbf{Top}(X, uY).$$

where uY is Y with the basepoint forgotten. The solution to this is X with a disjoint basepoint adjoined. Notation:

$$X/\varnothing = X_+.$$

We have another adjoint pair!

It's often useful to know that if $A \subseteq X$ and $B \subseteq Y$ then

$$(X/A) \wedge (Y/B) = \frac{X \times Y}{(A \times Y) \cup_{A \times B} (X \times B)}.$$

For example, if we think of $I^m/\partial I^m$ as our model of S^m as a pointed space, we find that

$$S^m \wedge S^n = (I^m/\partial I^m) \wedge (I^n/\partial I^n) = \frac{I^{m+n}}{(\partial I^m \times I^n) \cup (I^m \times \partial I^n)} = I^{m+n}/\partial I^{m+n} = S^{m+n}.$$

Smashing with S^1 is a critically important operation in homotopy theory, known as (reduced) *suspension*:

$$\Sigma X = S^1 \wedge X = \frac{I \times X}{(\partial I \times X) \cup (I \times *)}.$$

That is, the suspension is obtained from the cylinder by collapsing the top and the bottom to a point, as well as the line segment along a basepoint.

You are invited to check the various properties enjoyed by the smash product, analogous to properties of the tensor product. So it's functorial in both variables; the two-point pointed space serves as a unit; and it is associative and commutative. Associativity is a blessing bestowed by assuming compact generation; notice that in forming it we are mixing limits (the product) with colimits (the quotient by the axes), and indeed the smash product turns out *not* to be associative in the full category of spaces. By induction, the n-fold suspension is thus

$$\Sigma^n X = S^1 \wedge \Sigma^{n-1} X = S^1 \wedge (S^{n-1} \wedge X) = (S^1 \wedge S^{n-1}) \wedge X = S^n \wedge X.$$

The smash product and its adjoint render $k\mathbf{Top}_*$ a "closed symmetric monoidal category."

We can also think about the *loop space* of a pointed space,

$$\Omega X = X_*^{S^1},$$

or the iterated loop space $\Omega^n X$, which we claim equals $X_*^{S^n}$: by induction,

$$\Omega^n X = \Omega(\Omega^{n-1} X) = (X_*^{S^{n-1}})_*^{S^1} = X_*^{S^{n-1} \wedge S^1} = X_*^{S^n}.$$

You may be alarmed at the prospect of trying to understand the algebraic topology of a function space like ΩX. Perhaps the following theorem of John Milnor will be of some solace.

Theorem 3.3 (Milnor; see [9])**.** *If X is a pointed countable CW complex, then ΩX has the homotopy type of a pointed countable CW complex.*

The homotopy category

From now on, **Top** will mean k**Top**.

Formation of sets of homotopy classes of maps leads to a new category, the *homotopy category* (of spaces) Ho**Top**. The objects of Ho**Top** are the same as those of **Top**, but the set of morphisms from X to Y is given by $[X, Y]$. You should check that composition in **Top** descends to composition in Ho**Top**.

Be warned that the homotopy category has rather poor categorical properties. Products and coproducts in **Top** provide products and coproducts in Ho**Top**, but most other types of limits and colimits do not exist in Ho**Top**.

If we have basepoints around, we will naturally want our homotopies to respect them. A "pointed homotopy" between pointed maps is a function $h : I \times X \to Y$ such that $h(t, -)$ is pointed for all t. This means that it factors through the quotient of $I \times X$ obtained by pinching $I \times *$ to a point. This quotient space may be expressed in terms of the smash product:

$$\frac{I \times X}{I \times *} = I_+ \wedge X .$$

Pointed homotopy is again an equivalence relation, and we have the *pointed homotopy category*, or, more properly, the *homotopy category of pointed spaces* Ho**Top**$_*$. We'll write $[X, Y]_*$ for the set of maps in this category.

Definition 3.4. Let $(X, *)$ be a pointed space and n a positive integer. The nth *homotopy group* of X is

$$\pi_n(X) = [S^n, X]_* .$$

Note the long list of aliases for this set: for any k with $0 \le k \le n$,

$$\pi_n(X) = [S^n, X]_* = [S^0, \Omega^n X]_* = [S^k, \Omega^{n-k} X]_* = \pi_k(\Omega^{n-k} X) .$$

Since π_1 group-valued, $\pi_n(X)$ is indeed a group for any $n \ge 1$. These groups look innocuous, but they turn out to hold the solutions to many important geometric problems, and are correspondingly difficult to compute. For example, if a simply connected finite complex is not contractible then infinitely many of its homotopy are nonzero, and only finitely many of them are known.

4   Fiber bundles

Much of this course will revolve around variations on the following concept.

Definition 4.1. A *fiber bundle* is a map $p : E \to B$, such that for every $b \in B$, there exists an open subset $U \subseteq B$ containing b and a map $p^{-1}(U) \to p^{-1}(b)$ such that $p^{-1}(U) \to U \times p^{-1}(b)$ is a homeomorphism.

When $p : E \to B$ is a fiber bundle, E is called the *total space*, B the *base space*, and p the *projection*. The point pre-image $p^{-1}(b) \subseteq$ for $b \in B$ is the the *fiber over b*. We may use the symbol ξ for the bundle, and write $\xi : E \downarrow B$.

An *isomorphism* from $p : E \to B$ to $p' : E' \to B$ is a homeomorphism $f : E \to E'$ such that $p' \circ f = p$. The map $p : E \to B$ is a fiber bundle if it is "locally trivial," i.e. locally (in the base) isomorphic to a "trivial" bundle $\mathrm{pr}_1 : U \times F \to U$.

Fiber bundles are naturally occurring objects. For instance, a covering space $E \to B$ is precisely a fiber bundle with discrete fibers.

Example 4.2. The "Hopf fibration" provides a beautiful example of a fiber bundle. Let $S^3 \subset \mathbf{C}^2$ be the unit 3-sphere. Write $p : S^3 \to \mathbf{C}P^1 \cong S^2$ for the map sending a vector v to the complex line through v and the origin. This is a fiber bundle whose fiber is S^1.

We said "the fiber" of p is S^1. It's not hard to see that any two fibers of a fiber bundle over a path connected base space are homeomorphic, so this language isn't too bad. If we envision S^3 as the one-point compactification of $\mathbb{R}^3$, we can visualize how the various fibers relate to each other. The fiber through the point at infinity is a line in $\mathbb{R}^3$; imagine it as the z-axis. All the other fibers are circles. It's a great exercise to envision [15] how they fill up Euclidean space.

This map $S^3 \to S^2$ is the attaching map for the 4-cell in the standard CW structure on $\mathbb{C}P^2$. The nontriviality of the cup-square in $H^*(\mathbb{C}P^2)$ shows that it is *essential*, that is, not null-homotopic. This example is due to Heinz Hopf (1894–1971), a German mathematician working mainly at ETH in Zürich. He discovered the Hopf fibration and its nontriviality during a visit to Princeton in 1927–28. This was the first indication that spheres might have interesting higher homotopy groups.

Example 4.3. The *Stiefel manifold* $V_k(\mathbb{R}^n)$ is the space of orthogonal "k-frames," that is, ordered k-element orthonormal sets of vectors in $\mathbb{R}^n$. Equivalently, it is the space of linear isometric embeddings of $\mathbb{R}^k$ into $\mathbb{R}^n$; or the set of $n \times k$ matrices A such that $AA^T = I_k$. It is a compact manifold. (Eduard Stiefel (1909–1978) was a Swiss mathematician at ETH Zürich.)

We also have the *Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n)$, the space of k-dimensional vector subspaces of $\mathbb{R}^n$. (Hermann Grassmann (1809–1877) discovered much of the theory of linear algebra, but his work was not appreciated during his lifetime. He taught at a Gymnasium in Stettin, Poland, and wrote on linguistics.) By forming the span, we get a map

$$V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$$

generalizing the double cover $S^{n-1} \to \mathbb{R}P^{n-1}$ (which is the case $k = 1$). There is of course a complex analogue,

$$V_k(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$$

generalizing the Hopf bundle (which is the case $n = 2, k = 1$).

These maps are fiber bundles (with fiber over V given by the space of ordered orthonormal bases of V). We can regard fact this as a special case of the following general theorem about homogeneous spaces of compact Lie groups (such as $O(n)$, $U(n)$, or a finite group).

Proposition 4.4. *Let G be a compact Lie group and let $G \supseteq H \supseteq K$ a sequence of closed subgroups (also then compact Lie groups in their own right). Then the projection map between homogeneous spaces $G/K \to G/H$ is a fiber bundle.*

The orthogonal group $O(n)$ acts on the Stiefel manifold $V_k(\mathbb{R}^n)$ from the left, by postcomposition. This action is transitive, and the isotropy group of the basepoint is the subgroup $O(n - k) \times I_k \subseteq O(n)$. This means that

$$V_k(\mathbb{R}^n) = O(n)/O(n-k) \times I_k \,,$$

and we have a fibration $O(n) \to V_k(\mathbb{R}^n)$ with fiber $O(n-k)$. For example, $V_1(\mathbb{R}^n)$ is the unit sphere S^{n-1} in $\mathbb{R}^n$, so we have a fibration $O(n) \to S^{n-1}$ with fiber $O(n-1)$. This will be useful in an analysis of this topological group.

Another interesting map occurs if we forget all but the first vector in a k-frame. This gives us a map $V_k(\mathbb{R}^n) \to S^{n-1}$. This is the bundle of tangent $(k-1)$-frames on the $(n-1)$-sphere. A deep question asks for which n and k this bundle has a section.

The Grassmannian $\mathrm{Gr}_k(\mathbb{R}^n)$ is obtained by dividing by the larger subgroup $O(n-k) \times O(k)$, and Proposition 4.4 implies that the map $V_k(\mathbb{R}^n) \to \mathrm{Gr}_k(\mathbb{R}^n)$ is a fiber bundle.

Proposition 4.4 is a corollary of the following general criterion.

Theorem 4.5 (Ehresmann, 1951; see [8]). *Suppose E and B are smooth manifolds, and let $p : E \to B$ be a smooth (i.e., C^∞) map. If p is a* proper *(preimages of compact sets are compact) submersion (that is, $dp : T_e E \to T_{p(e)} B$ is a surjection for all $e \in E$), then it is a fiber bundle.*

Much of this course will consist of a study of fiber bundles such as these through various essentially algebraic lenses. To bring them into play, we will always demand a further condition of our bundles.

Definition 4.6. An open cover $\mathcal{U}$ of a space X is *numerable* if there exists subordinate partition of unity; i.e., there is a family of functions $\varphi_U : X \to [0,1] = I$, indexed by the elements of $\mathcal{U}$, such that $\varphi_U^{-1}((0,1]) = U$ and any $x \in X$ belongs to only finitely many $U \in \mathcal{U}$. The space X is *paracompact* if any open cover admits a numerable refinement. A fiber bundle is *numerable* if it admits a numerable trivializing cover.

So any fiber bundle over a paracompact space is numerable. This isn't too restrictive for us:

Proposition 4.7 (Miyazaki; see Theorem 1.3.5 in [9]). *CW-complexes are paracompact.*

5 Fibrations, fundamental groupoid

Fibrations and path liftings

During the 1940s, much effort was devoted to extracting homotopy-theoretic features of fiber bundles. It came to be understood that the desired consequences relied entirely on a "homotopy lifting property." One of the revolutions in topology around 1950 was the realization that it was advantageous to simply take that property as a *definition*. This extension of the notion of a fiber bundle included wonderful new examples, but still retained the homotopy theoretic consequences. Here is the definition.

Definition 5.1. A *fibration* is a map $p : E \to B$ that satisfies the *homotopy lifting property* ("HLP"): Given any $f : W \to E$ and any homotopy $h : I \times W \to B$ with $h(0,w) = pf(w)$, there is a map $\overline{h}$ that lifts h and extends f: that is, making the following diagram commute.

$$
\begin{array}{ccc}
W & \xrightarrow{\quad f \quad} & E \\
{\scriptstyle \mathrm{in}_0} \downarrow & {\overline{h} \nearrow} & \downarrow {\scriptstyle p} \\
I \times W & \xrightarrow{\quad h \quad} & B
\end{array}
\tag{1.1}
$$

For example, for any space X (even the empty space!) the unique map $X \to *$ is a fibration (A lift is given by $\overline{h}(t,w) = f(w)$.) as is the unique map $\varnothing \to X$ (Why?). In general, though, this seems like an alarming definition, since the HLP has to be checked for *all* spaces W, *all* maps f, and *all* homotopies h!

On the other hand, an advantage of this type of definition, by means of a lifting condition, is that it enjoys various easily checked persistence properties.

- Base change: If $p : E \to B$ is a fibration and $X \to B$ is any map, then the induced map $E \times_B X \to X$ is again a fibration. In particular, any product projection is a fibration.

- Products: If $p_i : E_i \to B_i$ is a family of fibrations then the product map $\prod p_i$ is again a fibration.

- Exponentiation: If $p : E \to B$ is a fibration and A is any space, then $E^A \to B^A$ is again fibration.

- Composition: If $p : E \to B$ and $q : B \to X$ are both fibrations, then the composite $qp : E \to X$ is again a fibration.

Not all of these persistence properties are true for fiber bundles. Which ones fail?

There is a nice geometric interpretation of what it means for a map to be a fibration, in terms of "path liftings". We'll use Cartesian closure! The adjoint of the solid arrow part of (1.1) is

$$
\begin{array}{ccc}
W & \xrightarrow{f} & E \\
\downarrow{\scriptstyle \widehat{h}} & & \downarrow{\scriptstyle p} \\
B^I & \xrightarrow{\text{ev}_0} & B
\end{array}
\tag{1.2}
$$

By the definition of the pullback, the data of this diagram is equivalent to a map $W \to B^I \times_B E$. Explicitly,

$$
B^I \times_B E = \{(\omega, e) \in B^I \times E : \omega(0) = p(e)\} \,.
$$

This space comes equipped with a map from E^I, given by sending a path $\omega : I \to E$ to

$$
\widetilde{p}(\omega) = (p\omega, \omega(0)) \in B^I \times_B E \,.
$$

In these terms, giving a lift $\overline{h}$ in (1.1) is equivalent to giving a lift

$$
\begin{array}{ccc}
 & & E^I \\
 & \nearrow & \downarrow{\scriptstyle \widetilde{p}} \\
\widetilde{h} & & \\
W & \longrightarrow & B^I \times_B E
\end{array}
$$

This again needs to be checked for every W and every map to $B^I \times_B E$. But at least there is now a universal case to consider: $W = B^I \times_B E$ mapping by the identity map! So p is a fibration if and only if a lift λ exists in the following diagram; that is, a *section* of $\widetilde{p}$:

$$
\begin{array}{ccc}
 & & E^I \\
 & \nearrow & \downarrow{\scriptstyle \widetilde{p}} \\
\lambda & & \\
B^I \times_B E & \xrightarrow{\ 1\ } & B^I \times_B E
\end{array}
$$

The section λ is called a *path lifting function*. To understand why, suppose $(\omega, e) \in B^I \times_B E$, so that ω is a path in B with $\omega(0) = p(e)$. Then $\lambda(\omega, e)$ is then a path in E lying over ω and starting at e. The path lifting function provides a continuous lift of paths in B. The existence (or not) of a section of $\widetilde{p}$ provides a single condition that needs to be checked if you want to see that p is a fibration.

There is no mention of local triviality in this definition. However:

Theorem 5.2 (Albrecht Dold, 1963; see [42], Chapter 13)**.** *Let $p : E \to B$ be a continuous map. Assume that there is a numerable cover of B, say $\mathcal{U}$, such that for every $U \in \mathcal{U}$ the restriction $p|_{p^{-1}(U)} : p^{-1}U \to U$ is a fibration. Then p itself is a fibration.*

Corollary 5.3. *Any numerable fiber bundle is a fibration.*

Comparing fibers over different points

If $p : E \to B$ is a covering space, then unique path lifting provides, for any path ω from a to b, a homeomorphism $F_a \to F_b$ depending only on the path homotopy class of ω. Our next goal is to construct an analogous map for a general fibration.

Consider the solid arrow diagram:

$$
\begin{array}{ccc}
F_a & \longrightarrow & E \\
\downarrow{\scriptstyle \text{in}_0} & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\
I \times F_a & \underset{\text{pr}_1}{\longrightarrow} I \underset{\omega}{\longrightarrow} & B \,.
\end{array}
$$

This commutes since $\omega(0) = a$. By the homotopy lifting property, there is a dotted arrow that makes the entire diagram commute. If $x \in F_a$, the image $h(1, x)$ is in F_b. This supplies us with a map $f : F_a \to F_b$, given by $f(x) = h(1, x)$.

Since we are not working with a covering space, there will in general be many lifts h and so many choices of f. But we may at least hope that the homotopy class of f is determined by the path homotopy class of ω.

So suppose we have two paths ω_0, ω_1, with $\omega_0(0) = \omega_1(0) = a$ and $\omega_0(1) = \omega_1(1) = b$, and a homotopy $g : I \times I \to B$ between them (so that $g(0, t) = \omega_0(t)$, $g(1, t) = \omega_1(t)$, $g(s, 0) = a$, $g(s, 1) = b$). Here's a picture.

$$
\begin{array}{c}
b \\
t \uparrow \quad \omega_0 \; \boxed{\quad g \quad} \; \omega_1 \\
a \\
\underset{s}{\longrightarrow}
\end{array}
$$

Choose lifts h_0 and h_1 as above. These data are captured by a diagram of the form

$$
\begin{array}{ccc}
((\partial I \times I) \cup (I \times \{0\})) \times F_a & \longrightarrow & E \\
\downarrow{\scriptstyle \text{in}_0} & \nearrow & \downarrow{\scriptstyle p} \\
I \times I \times F_a & \underset{\text{pr}_1}{\longrightarrow} I \times I \underset{g}{\longrightarrow} & B
\end{array}
$$

The map along the top is given by h_0 and h_1 on $\partial I \times I \times F_a$ and by $\text{pr}_2 : I \times F_a \to F_a$ followed by the inclusion on the other summand.

If the dotted lift exists, it would restrict on $I \times \{1\} \times F_a$ to a homotopy between f_0 and f_1. Well, the subspace $(\partial I \times I) \cup (I \times \{0\})$ of $I \times I$ wraps around three edges of the square. It's easy enough to create a homeomorphism with the pair $(I \times I, \{0\} \times I)$, so the HLP (with $W = I \times F_a$) gives us the dotted lift.

So the map $F_a \to F_b$ is well-defined up to homotopy by the path homotopy class of the path ω from a to b. Let's denote it by f_ω.

The fundamental groupoid

We can set this up in categorical terms. The space B defines a category whose objects are the points of B and in which a morphism from a to b is a homotopy class of paths from a to b. Composition

is given by the juxtaposition rule

$$(\sigma \cdot \omega)(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \sigma(2t - 1) & 1/2 \le t \le 1. \end{cases}$$

The constant path c_a serves as an identity at up to homotopy: here are pictures of the homotopy between $c_b \cdot \omega$ and ω, and between $\sigma \cdot c_a$ and σ.

Similar pictures show that $(\alpha \cdot \sigma) \cdot \omega \simeq \alpha \cdot (\sigma \cdot \omega)$ and that every morphism has an inverse, given by $\overline{\omega}(t) = \omega(1 - t)$.

This gives us a *groupoid* – a small category in which every morphism is an isomorphism – called the *fundamental groupoid* of B, and written with a capital π: $\Pi_1(X)$.

Our work can be succinctly summarized as follows.

Proposition 5.4. *Formation of fibers of a fibration $p : E \to B$ determines a functor $\Pi_1(B) \to$* Ho**Top**.

Proof. We should check functoriality: if $\omega : a \sim b$ and $\sigma : b \sim c$, then hopefully the induced homotopy classes compose:

$$f_{\sigma\omega} = f_\sigma \circ f_\omega.$$

To see this, pick lifts h_ω and h_σ in

so that $f_\omega(e) = h_\omega(1, e)$ and $f_\sigma(e) = h_\sigma(1, e)$. Then construct a lifting in
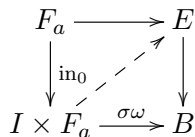
by using h_ω in the left half of the interval and $h_\sigma \circ f_\omega$ in the right half. The resulting map $F_a \to F_b$ is then precisely $f_\sigma \circ f_\omega$. $\square$

Remark 5.5. Last semester we defined the product of loops as juxtaposition but in the reverse order. That convention would have produced a contravariant functor $\Pi_1(X) \to$ Ho**Top**.

Remark 5.6. Since any functor carries isomorphisms to isomorphisms, Proposition 5.4 implies that a path from a to b determines a homotopy class of homotopy equivalences from F_a to F_b.

Fix a map $p : E \to Y$. The pullback of E along a map $f : X \to Y$ can vary wildly as f is deformed; it is far from being a homotopy invariant. Just think of the case $X = *$, for example, when the pullback along $f : * \to Y$ is the point preimage $p^{-1}(f(*))$. One of the great features of fibrations is this:

Proposition 5.7. *Let $p : E \to Y$ be a fibration and $f_0, f_1 : X \to Y$ two maps. Write E_0 and E_1 for pullbacks of E along f_0 and f_1. If f_0 and f_1 are homotopic then E_0 and E_1 are homotopy equivalent.*

Proof. We construct a fibration over Y^X whose fiber over f is f^*E, the pullback of $E \to Y$ along f. It occurs as the middle vertical composite in the following diagram of pullbacks.

$$
\begin{array}{ccccc}
f^*E & \longrightarrow & E \times_Y (Y^X \times X) & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
* \times X & \xrightarrow{\ \mathrm{in}_f \times 1\ } & Y^X \times X & \xrightarrow{\ \mathrm{ev}\ } & Y \\
\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \\
* & \xrightarrow{\ \mathrm{in}_f\ } & Y^X & &
\end{array}
$$

The middle horizontal composite is the map f, so the pullback is f^*E as shown. Now a homotopy between f_0 and f_1 is a path in Y^X from f_0 to f_1, and so by Lemma 5.4 the fibers over them are homotopy equivalent.   □

Remark 5.8. We could ask for more: We could ask that E_0 and E_1 are homotopy equivalent by maps and homotopies respecting the projections to X: that there is a *fiber homotopy equivalence* between them. This is in fact the case, as you will show for homework.

Corollary 5.9. *Let $p : E \to B$ be a fibration. If B is contractible to $* \in B$, then the inclusion of the fiber $p^{-1}(*) \hookrightarrow E$ is a homotopy equivalence.*

Proof. The identity map 1_B and the constant map $c : B \to B$ with value $*$ are homotopic, so pulling back $E \downarrow B$ along them produce homotopy equivalent spaces. One gives E, the other $B \times p^{-1}(*)$. The projection $\mathrm{pr}_2 : B \times p^{-1}(*) \to p^{-1}(*)$ is a homotopy equivalence since B is contractible. We leave you to check that the resulting equivalence is the inclusion.   □

6   Cofibrations

Let $i : A \to X$ be a map of spaces, and Y some other space. When is the induced map $Y^X \to Y^A$ a fibration? For example, if $a \in X$, does evaluation at a produce a fibration $Y^X \to Y$?

By the definition of a fibration, we want a lifting in the solid-arrow diagram

$$
\begin{array}{ccc}
W & \longrightarrow & Y^X \\
\downarrow{\scriptstyle \mathrm{in}_0} & \nearrow & \downarrow \\
I \times W & \longrightarrow & Y^A
\end{array}.
$$

Adjointing over, we get:

$$
\begin{array}{ccc}
A \times W & \xrightarrow{\ i \times 1\ } & X \times W \\
\downarrow{\scriptstyle 1 \times \mathrm{in}_0} & & \downarrow \\
A \times I \times W & \longrightarrow & X \times I \times W \\
& \searrow & \searrow \\
& & Y\,.
\end{array}
$$

Adjointing over again, this diagram transforms to:

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
A \times I & \longrightarrow & X \times I \\
& \searrow & \searrow \\
& & Y^W\,.
\end{array}
$$

This discussion motivates the following definition of a cofibration, "dual" to the notion of fibration.

Definition 6.1. A *cofibration* is a map $i : A \to X$ that satisfies *homotopy extension property* (sometimes abbreviated as "HEP"): for any solid-arrow commutative diagram as below, a dotted arrow exists making the whole diagram commutative.

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & X \\
\downarrow & & \downarrow \\
A \times I & \longrightarrow & X \times I \\
& \searrow & \searrow \\
& & Z\,.
\end{array}
$$

How shall we check that a map is a cofibration? By the universal property of a pushout, $A \to X$ is a cofibration if and only if there is an extension in

$$
\begin{array}{ccc}
(X \times 0) \cup_A (A \times I) & \xrightarrow{\ j\ } & X \times I \\
& \searrow{\scriptstyle f} & \vdots \\
& & Z
\end{array}
$$

for every map f. Now there is a universal example, namely $Z = (X \times 0) \cup_A (A \times I)$, $f = \mathrm{id}$. So a map i is a cofibration if and only if the map $j : (X \times 0) \cup_A (A \times I) \to X \times I$ admits a retraction: a map $r : X \times I \to (X \times 0) \cup_A (A \times I)$ such that $rj = 1$.

The space involved is called the *mapping cylinder*, and written

$$
M(i) = (X \times 0) \cup_A (A \times I)\,.
$$

It's not hard to check (using the mapping cylinder) that any cofibration is a subspace embedding. But the map j may not be an embedding; the map $(X \times 0) \cup_A (A \times I) \to \mathrm{im}(j) \subseteq X \times I$ is a continuous bijection but it may not be a homeomorphism. If $A \subseteq X$ is a *closed* subset then $A \times I \subseteq X \times I$ is

a closed map, and $X \times 0 \subseteq X \times I$ is also, so the map from the pushout is a closed map and hence is then a homeomorphism to its image.

So the inclusion of a closed subspace $A \subseteq X$ is a cofibration if and only if there is a retraction from $X \times I$ onto its subspace $(X \times 0) \cup (A \times I)$.

Definition 6.2. A basepoint $*$ in X is *nondegenerate* if $\{*\} \hookrightarrow X$ is a closed cofibration. One also says that $(X, *)$ is *well-pointed.*

Any point in a CW complex, for example, will serve as a nondegenerate basepoint. If $*$ is a nondegenerate basepoint of A, the evaluation map ev $: X^A \to X$ is a fibration, The fiber of ev over the basepoint of X is then exactly the space of pointed maps X_*^A. Whenever convenient we will assume our basepoints are nondegenerate.

Example 6.3. $i : S^{n-1} \hookrightarrow D^n$ is a cofibration: The map

$$j : D^n \cup_{S^{n-1}} (S^{n-1} \times I) \hookrightarrow D^n \times I$$

is the inclusion of the open tin can into the closed can full of soup –
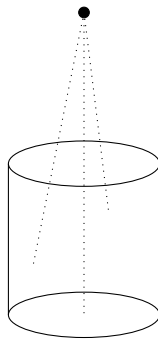
The illustrated retraction of $D^n \times I$ onto the open can send a point in the soup to its shadow on the open tin can.

In particular, setting $n = 1$ in this example, $\{0, 1\} \hookrightarrow I$ is a cofibration, so evaluation at the pair of endpoints,

$$\mathrm{ev}_{0,1} : Y^I \to Y \times Y \,,$$

is a fibration. Every point in I is nondegenerate, so $ev_a : Y^I \to Y$ is a fibration for any $a \in I$.

The class of cofibrations is closed under the following operations.

- Cobase change: if $A \to X$ is a cofibration and $A \to B$ is any map, the pushout $B \to X \cup_A B$ is again a cofibration.

- Coproducts: if $A_j \to X_j$ is a cofibration for every j, then the coproduct map $\coprod A_j \to \coprod X_j$ is again a cofibration.

- Product: If $A \to X$ is a cofibration and B is any space, then $A \times B \to X \times B$ is again a cofibration.

- Composition: If $A \to B$ and $B \to X$ are both cofibrations, then the composite $A \to X$ is again a cofibration.

It follows from these inheritance properties and the single example $S^{n-1} \hookrightarrow D^n$ that if X is a CW complex and A is a subcomplex then $A \to X$ is a cofibration. CW complexes are Hausdorff spaces, and in any such a case a cofibration is a closed embedding.

Cofibrance provides a natural condition under which a contractible subspace can be collapsed with out damage.

Proposition 6.4. *Let $A \to X$ be a cofibration, and write X/A for the pushout of $* \leftarrow A \to X$. If A is contractible then $X \to X/A$ is a homotopy equivalence.*

Proof. Pick a contracting homotopy $h : A \times I \to A$, so that $h(a,0) = a$ and $h(a,1) = * \in A$ for all $a \in A$. By cofibrance there is an extension of $f \circ h$ to a homotopy $g : X \times I \to X$ such that $g(x,0) = x$. $g(-,1)$ then factors through the projection $p : X \to X/A$: there is a map $r : X/A \to X$ such that $r \circ p$ is homotopic the identity.

To construct a homotopy from $p \circ r$ to $1 : X/A \to X/A$, note that the homotopy g sends $A \times I$ into A, so its composite with $p : X \to X/A$ factors through a map $\bar{g} : (X/A) \times I \to X/A$. At $t = 0$ this is the identity; at $t = 1$ it is just $p \circ r$. $\qquad\square$

7   Cofibration sequences and co-exactness

There is a pointed version of the cofibration condition: but you only ask to extend *pointed* homotopies; so the condition is weaker than the unpointed version. (It's true that we seek an extension to a *pointed homotopy*, but since the basepoint is in the source space this is automatic.) A pointed homotopy can be thought of as a pointed map

$$X \wedge I_+ = \frac{X \times I}{* \times I} \to Y$$

The condition that the embedding of a closed subspace $i : A \subseteq X$ is a pointed cofibration can again be expressed as requiring that the inclusion of the (now "reduced") mapping cylinder

$$M(i) = (X \times 0) \cup_{A \times 0} (A \wedge I_+)$$

into $X \wedge I_+$ admits a retraction. Today we'll work entirely in the pointed context, and I'll tend to omit the adjectives "reduced" and "pointed." (Maybe I should have written M^u for the unpointed variant!)

Any pointed map $f : X \to Y$ admits a canonical factorization as a closed pointed cofibration followed by a pointed homotopy equivalence:

where i embeds X along $t = 1$. For example, the *cone* on a space X is a mapping cylinder:

$$CX = M(X \to *) = X \wedge I \,.$$

The map $X \to *$ factors as the cofibration $X \to CX$ followed by the homotopy equivalence $CX \to *$.

Since i is a cofibration, we should feel entitled to collapse it to a point; that is, form the pushout in

$$\begin{array}{ccc} & X & \longrightarrow & * \\ {\scriptstyle f}\nearrow & \downarrow{\scriptstyle i} & & \downarrow \\ Y & \overset{\simeq}{\longleftarrow} M(f) & \longrightarrow & C(f) \,. \end{array}$$

$C(f)$ is the *mapping cone* of f. If the mapping cylinder is a top hat, the mapping cone is a witch's hat. One example: the suspension functor is given by

$$\Sigma X = C(X \to *) \,.$$

Since i is a cofibration, the pushout $* \to C(f)$ is again a cofibration; the cone point is always nondegenerate.

This pushout can be expressed differently: Instead of replacing $f : X \to Y$ with a cofibration, let's replace $X \to *$ with a cofibration, namely, the inclusion $X \hookrightarrow CX$. So we have a pushout diagram

$$\begin{array}{ccc} & & * \\ & \nearrow & \uparrow{\scriptstyle \simeq} \\ X & \overset{\mathrm{in}_1}{\longrightarrow} & CX \\ \downarrow{\scriptstyle f} & & \downarrow \\ Y & \overset{i(f)}{\longrightarrow} & C(f) \,. \end{array}$$

This pushout is homeomorphic to the earlier one; but notice that the homeomorphism uses the automorphism of the unit interval sending t to $1 - t$.

If f is already a cofibration, the cobase change property implies that $CX \to C(f)$ is again cofibration. CX is contractible, so by Proposition 6.4, collapsing it to a point is a homotopy equivalence. But collapsing CX in $C(f)$ is the same as collapsing Y in X:

Lemma 7.1. *If $f : X \to Y$ is a cofibration then the collapse map $C(f) \to Y/X$ is a homotopy equivalence.*

Co-exactness

Definition 7.2. A *cofibration sequence* is a diagram that that is homotopy equivalent to

$$X \overset{f}{\to} Y \overset{i(f)}{\longrightarrow} C(f)$$

for some map f.

The composite $X \to C(f)$ is *null-homotopic*; that is, it's homotopic to the constant map (with value the basepoint). The homotopy is given by $h : (x, t) \mapsto [x, t]$: When $t = 0$ we can use $[x, 0] \sim f(x)$ to see the composite, while when $t = 1$ we get the constant map.

The pair $(i(f), h)$ is *universal* with this property: giving a map $g : Y \to Z$ along with a null-homotopy of the composite $g \circ f$ is the same thing as giving a map $C(f) \to Z$ that extends g.

An implication of this is the following:

Lemma 7.3. *For any pointed map $f : X \to Y$ and any pointed space Z, the sequence of pointed sets*

$$[X, Z]_* \overset{f^*}{\longleftarrow} [Y, Z]_* \overset{i(f)^*}{\longleftarrow} [C(f), Z]_*$$

is exact, *in the sense that*

$$\text{im}(i(f)^*) = \{g : Y \to Z : g \circ f \simeq *\}\,.$$

Any sequence of composable arrows with this property is "*co-exact*": so cofibration sequences are coexact.

The map $f : X \to Y$ functorially determines the map $i(f) : Y \to C(f)$, and we may form *its* mapping cone, and continue:

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C(f) \xrightarrow{i^2(f)} C(i(f)) \xrightarrow{i^3(f)} C(i^2(f)) \xrightarrow{i^4(f)} \cdots .$$

This looks like it will lead off into the wilderness, but luckily there is a kind of periodicity at work. Here's a picture of $C(i(f))$:
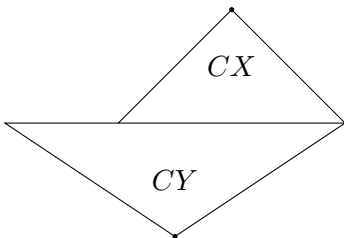
The map $i(f)$ is the pushout of the cofibration $X \to CX$ along $X \to Y$, so it is a cofibration. Therefore, by Lemma 7.1 the collapse map $C(i(f)) \to C(f)/Y$ is a homotopy equivalence. But

$$C(f)/Y = \Sigma X\,,$$

the suspension of X. So we have the commutative diagram

Now we have two ways to continue! I combine them in the homotopy commutative diagram

Notice the minus sign! It means that instead of $[t, x] \mapsto [t, f(x)]$, we have to use $[t, x] \mapsto [1-t, f(x)]$. This is needed to make the triangle commute, even up to homotopy, as you can see by being careful with the parametrization of the cones.

The resulting long sequence of maps

$$X \to Y \to C(f) \to \Sigma X \to \Sigma Y \to \Sigma C(f) \to \Sigma^2 X \to \cdots$$

is the *Barratt-Puppe sequence* associated to the map f. Each two-term subsequence is a cofiber sequence and is co-exact.

The Barratt-Puppe sequence is a "homotopy theoretic" version of the long exact homology sequence of a pair. Suppose that A is a subspace of X. Then I claim that

$$\overline{H}_*(X \cup CA) \cong H_*(X, A)$$

If you combine that with the suspension isomorphism in reduced homology, the Barratt-Puppe sequence gives you the homology long exact sequence of the pair.

To see the equality, just use homotopy invariance and excision:

$$\overline{H}_*(X \cup CA) = H_*(X \cup CA, *) = H_*(X \cup CA, CA)$$

$$= H_*(X \cup C_{\leq(1/2)}A, C_{\leq(1/2)}A) = H_*(X \cup A \times I, A \times I) = H_*(X, A).$$

Since $X \cup CA \simeq X/A$ if $A \to X$ is a cofibration, this is a good condition to guarantee that

$$H_*(X, A) = \overline{H}_*(X/A).$$

8 Weak equivalences and Whitehead's Theorems

We now have defined the homotopy groups of a pointed space,

$$\pi_n(X) = [S^n, X]_*.$$

So $\pi_0(X)$ is the pointed set of path components. For $n > 0$, π_n only sees the path component of the basepoint. It's a group for $n = 1$, and hence also for $n \geq 1$ since $\pi_n(X) = \pi_1(\Omega^{n-1})$.

Here's another very useful way to represent an element of $\pi_n(X, *)$. Recall our description of the n-sphere as a pointed space:

$$S^n = I^n/\partial I^n.$$

So an element of $\pi_n(X, *)$ is a homotopy class of maps of pairs

$$(I^n, \partial I^n) \to (X, *).$$

Lemma 8.1. *For $n \geq 2$, $\pi_n(X)$ is abelian.*

Proof. I'll give you two proofs of this fact. Since $\pi_n(X) = \pi_2(\Omega^{n-2}X)$, it suffices to consider $n = 2$.

First, geometric: Given $f, g : I^2 \to X$, both sending ∂I^2 to $*$, we can form another one by putting the two side by side (and compressing the horizontal coordinate by a factor of 2 in each). This is the sum in $\pi_n(X)$. This is homotopic to the map that does f and g in much smaller rectangles and fills in the rest of the square with maps to the basepoint. Now I'm free to move these two smaller rectangles around one another, exchanging positions. Then I can re-expand, to get the addition $g + f$.
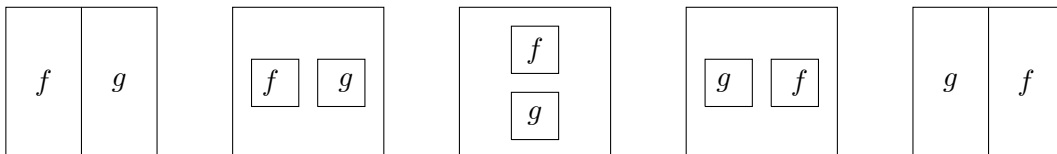
Now, algebraic: An *H-space* is a pointed space Y together with map $\mu : Y \times Y \to Y$ such that

$$Y \xrightarrow{\text{in}_1} Y \times Y \xleftarrow{\text{in}_2} Y$$

with maps 1, μ, 1 to Y

commutes in $\text{Ho}(\mathbf{Top}_*)$. The relevant example here is $Y = \Omega X$. Then $\pi_1(Y, *)$ has extra structure: Since $\pi_1(Y \times Y, *) = \pi_1(Y, *) \times \pi_1(Y, *)$ (as groups) we get a group G together with a group homomorphism $\mu : G \times G \to G$ such that

$$G \xrightarrow{\text{in}_1} G \times G \xleftarrow{\text{in}_2} G$$

with maps 1, μ, 1 to G

commutes. That is to say, $\mu(a, 1) = a$, $\mu(1, d) = d$, and, since $(a, b) \cdot (c, d) = (ac, bd)$ in $G \times G$,

$$\mu(ac, bd) = \mu(a, b) \cdot \mu(c, d)\,.$$

Take $b = 1 = c$ so $\mu(a, d) = ad$: that is, the "multiplication" μ is none other than the group multiplication. Then take $a = 1 = d$ so $\mu(c, b) = bc$: that is, the group structure is commutative. □

We can trace what happens when we move the basepoint. Let $\omega : I \to X$ be a path from a to b. It induces a map

$$\omega_\# : \pi_n(X, a) \to \pi_n(X, b)$$
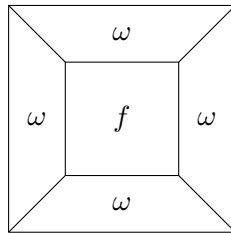
in the following way. Given $f : I^n \to X$ representing $\alpha \in \pi_n(X, *)$, define a map

$$(I^n \times 0) \cup (\partial I^n \times I) \to X$$

by

$$(v, t) \mapsto \begin{cases} f(v) & \text{for } v \in I^n, t = 0 \\ \omega(t) & \text{for } v \in \partial I^n\,. \end{cases}$$

Precompose this map with the map from the face $I^n \times 1$ given by projecting from the point $(b, 2)$, where b is the center of I^n. The result is a new map $I^n \to X$; it sends the middle part of the cube by f, and the peripheral part by ω.

It's easy to check that this gives rise to a functor $\Pi_1(X) \to \mathbf{Set}$, and hence to an action of $\pi_1(X, *)$ on $\pi_n(X, *)$. For $n = 1$, this is the conjugation action,

$$\omega \cdot \alpha = \omega \alpha \omega^{-1}\,.$$

For all $n \geq 1$ it is an action by group homomorphisms; for $n \geq 2$, $\pi_n(X, *)$ is a $\mathbb{Z}[\pi_1(X, *)]$-module.

Definition 8.2. A space is *simple* if this action is trivial for every choice of basepoint.

Example 8.3. If all path components are simply connected, the space is simple. A topological group is a simple space.

This action can be used to explain how homotopic maps act on homotopy groups.

Proposition 8.4. *Let $h : f_0 \sim f_1$ be a ("free," as opposed to pointed) homotopy of maps $X \to Y$. Let $* \in X$, and let $\omega : I \to X$ by $\omega(t) = h(*, t)$. Then*

$$
\begin{array}{ccc}
 & & \pi_n(Y, f_0(*)) \\
 & \overset{f_{0*}}{\nearrow} & \Big\downarrow {\scriptstyle \omega_\#} \\
\pi_n(X, *) & & \\
 & \underset{f_{1*}}{\searrow} & \\
 & & \pi_n(Y, f_1(*))
\end{array}
$$

commutes.

Proof. The homotopy h fills in the cube $I^n \times I$, and provides a pointed homotopy from $\omega \cdot f_0$ to f_1. $\qquad\square$

While it may be hard to compute homotopy groups, we can think about what sort of maps induce isomorphisms in them.

Definition 8.5. A map $f : X \to Y$ is a *weak equivalence* if it induces an isomorphism in π_0 and in π_n for all $n \geq 1$ and every choice of basepoint in X.

Of course it suffices to pick one point in each path component.

Weak equivalences may not have any kind of map going in the opposite direction. The definition seems very base-point focused, but in fact it is not. For example,

Proposition 8.6. *Any homotopy equivalence is a weak equivalence.*

Proof. Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$, and pick a homotopy $h : 1_X \sim gf$. Define $\omega : I \to X$ by $\omega(t) = h(*, t)$. Then by Proposition 8.4 we have a commutative diagram

$$
\begin{array}{ccc}
\pi_n(X, *) & \overset{f_*}{\longrightarrow} & \pi_n(Y, f(*)) \\
 & \underset{\omega_\#}{\searrow} & \Big\downarrow {\scriptstyle g_*} \\
 & & \pi_n(X; gf(*))
\end{array}
$$

in which the diagonal is an isomorphism. Picking a homotopy $1 \sim fg$ gives the rest of the diagram

$$
\begin{array}{ccccc}
\pi_n(X, *) & \overset{f_*}{\longrightarrow} & \pi_n(Y, f(*)) & & \\
 & {\scriptstyle \cong}\searrow & \Big\downarrow {\scriptstyle g_*} & \searrow {\scriptstyle \cong} & \\
 & & \pi_n(X, gf(*)) & \overset{f_*}{\longrightarrow} & \pi_*(Y, fgf(*)) \,.
\end{array}
$$

It follows that g_* is an isomorphism, and therefore f_* is also. $\qquad\square$

Here are three fundamental theorems about weak equivalences, all due more or less to J.H.C. Whitehead. (John Henry Constantine Whitehead (fl. 1930–1961, Oxford) was a pioneer in the development of homotopy theory, inventor i.a. of CW complexes.)

Theorem 8.7. *Any weak equivalence induces an isomorphism in singular homology.*

Since $H_0(X)$ is the free abelian group generated by $\pi_0(X)$, this is obvious in dimension 0, and on each path component Poincaré's theorem implies it in dimension 1.

Theorem 8.8. *Let X and Y be simple spaces. Any map from X to Y that induces an isomorphism in homology is a weak equivalence.*

Theorem 8.9. *Let X and Y be CW complexes. Any weak equivalence from X to Y is in fact a homotopy equivalence.*

Theorem 8.8 clearly provides a powerful way to construct weak equivalences, and, when combined with Theorem 8.9, homotopy equivalences. We will prove a vast generalization of Theorem 8.8 later in the course.

Here is a useful strengthening of Theorem 8.9:

Theorem 8.10 ("Whitehead's little theorem"). *A map $f : X \to Y$ is a weak equivalence if and only if $f \circ - : [W, X] \to [W, Y]$ is bijective for all CW complexes W.*

Proof of 8.10⇒8.9. We assume that

$$f \circ - : [K, X] \to [K, Y]$$

is bijective for every CW complex K. Taking $K = Y$, we find that there is a map $g : Y \to X$ such that $f \circ g = 1_Y$. We claim that $g \circ f = 1_X$ as well. To see this we take $K = X$: so

$$f \circ - : [X, X] \to [X, Y]$$

is a monomorphism. Under it $1 \mapsto f$, but $g \circ f$ does as well:

$$g \circ f \mapsto f \circ (g \circ f) = (f \circ g) \circ f = 1_Y \circ g = f \,.$$

So $g \circ f = 1_X$.   □

Remark 8.11. There is a deep shift of focus involved here. In the beginning, homotopy theory dealt with what happens when you define an equivalence relation ("homotopy") on maps. Focusing on weak equivalences is an entirely different perspective: we are picking out a collection of maps that will be regarded as "equivalences." They are to become the isomorphism in the homotopy category. The fact that they satisfy 2-out-of-3 makes the collection of weak equivalences an appropriate choice.

This change in perspective may be attributed to Daniel Quillen, who, in *Homotopical Algebra* (written while Quillen was a professor at MIT, in collaboration with his colleague Dan Kan), set out an axiomatization of homotopy theory using three classes of maps, which he termed "weak equivalences," "cofibrations," and "fibrations." They are assumed to be related to each other through appropriate factorization and lifting properties. The resulting theory of "model categories" dominated the underlying framework of homotopy theory for thirty years, and is still a critically important tool.

9 Homotopy long exact sequence and homotopy fibers
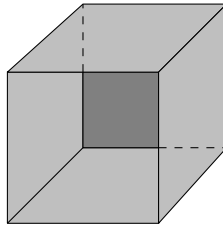
Relative homotopy groups

We'll continue to think of $\pi_n(X, *)$ as a set of homotopy classes of maps of pairs:

$$\pi_n(X, *) = [(I^n, \partial I^n), (X, *)].$$

As usual in algebraic topology, there is much to be gained from establishing a "relative" version. We will use the sequence of subspaces

$$I^n \supseteq \partial I^n \supseteq \partial I^{n-1} \times I \cup I^{n-1} \times 0$$

in this definition. We will write J_n for the last subspace, so for example $J_1 = \{0\} \subset I$. In general it's an "open box":

Definition 9.1. Let $(X, A, *)$ be a pointed pair. For $n \geq 1$, define a pointed set

$$\pi_n(X, A, *) = [(I^n, \partial I^n, J_n), (X, A, *)].$$

This definition is set up in such a way that

$$\pi_n(X, \{*\}, *) = \pi_n(X, *)$$

so that the inclusion $\{*\} \hookrightarrow A$ induces a map

$$\pi_n(X, *) \to \pi_n(X, A, *).$$

Also, restricting to the "back face" $I^{n-1} \times 0$ provides a map

$$\partial : \pi_n(X, A, *) \to \pi_{n-1}(A, *)$$

and the composite of these two is obviously "trivial," meaning that its image is the basepoint $* \in \pi_{n-1}(A, *)$. We get a sequence of pointed sets

$$
\begin{array}{ccccc}
& & \cdots \longrightarrow & \pi_3(X, A, *) & \\
& & {\scriptstyle \partial} \swarrow & & \\
\pi_2(A, *) \longrightarrow & \pi_2(X, *) \longrightarrow & \pi_2(X, A, *) & & \\
& & {\scriptstyle \partial} \swarrow & & \\
\pi_1(A, *) \longrightarrow & \pi_1(X, *) \longrightarrow & \pi_1(A, X, *) & & \\
& & {\scriptstyle \partial} \swarrow & & \\
\pi_0(A, *) \longrightarrow & \pi_0(X, *) & & &
\end{array}
$$

We claim that this is an exact sequence of pointed sets: the *long exact homotopy sequence of a pair*. For example, an element of $\pi_1(X, A, *)$ is represented by a path starting at the basepoint and ending in A. Its boundary is the component of that point in A. Saying that the component of $a \in A$ maps to the base point component of X is exactly saying that $[a] \in \pi_0(A)$ is in the image of $\partial : \pi_1(X, A, *) \to \pi_0(A, *)$.

We will investigate the structure of these relative homotopy groups, and explain why the sequence is exact, by developing an analogue of the Barratt-Puppe sequence that will turn out to give rise to the homotopy long exact sequence of a pair.

Fiber sequences

In the pointed category, we could redefine "fibration" slightly (as is done in [21], for example) so that $p : E \to B$ is a fibration if every pointed solid arrow diagram

$$
\begin{array}{ccc}
W & \longrightarrow & E \\
\downarrow{\scriptstyle \mathrm{in}_0} & \nearrow & \downarrow{\scriptstyle p} \\
I_+ \wedge W & \longrightarrow & B
\end{array}
$$

admits a lift. There are fewer diagrams, but more is demanded of the lift.

Instead we'll leave the fibrations as they are, but in compensation insist that our basepoints should be nondegenerate. Lifting is then contained in the following lemma. See [40] for the proof, which we forgo, preferring to give the proof of similar result 10.4 later.

Lemma 9.2 (Relative homotopy lifting property). *Let $A \subseteq X$ be a closed cofibration and $E \to B$ a fibration. Then a lifting exists in any solid arrow diagram*

$$
\begin{array}{ccc}
(X \times 0) \cup (A \times I) & \longrightarrow & E \\
\downarrow & \nearrow & \downarrow \\
X \times I & \longrightarrow & B\,.
\end{array}
$$

Exactly the same proof we did before shows that if $A \to B$ is a pointed cofibration and the basepoints are nondegenerate then $X_*^B \to X_*^A$ is a fibration. For example we can take $(B, A, *) = (I, \partial I, 0)$ to see that the map from the path space

$$
P(X) = X_*^I = \{\omega : I \to X : \omega(0) = *\}
$$

to X by evaluation at 1 is a fibration.

Taking A to be a singleton in Lemma 9.2:

Corollary 9.3. *Let $p : E \to B$ be a fibration and suppose given $f : W \to E$ and $g : W \to B$ such that $pf \simeq g$: so g is a lift of f up to homotopy. Then f is homotopic to a lift "on the nose," that is, a function $\overline{f} : W \to E$ such that $p\overline{f} = g$.*

So if $g : W \to B$ is such that $pg \simeq *$, then g is homotopic to a map that lands in the fiber $p^{-1}(*) = F$ of p over $*$. This shows that the sequence – the "fiber sequence" – of pointed spaces

$$
F \to E \to B
$$

is "exact," in the sense that for any well-pointed space W the sequence

$$[W, F]_* \to [W, E]_* \to [W, B]_*$$

is exact.

Not every map is a fibration, but every map factors as

$$X \xrightarrow{\simeq} T(f) \quad = \quad \{(x, \omega) \in X \times Y^I : \omega(1) = f(x)\}$$

where $X \to T(f)$ is a homotopy equivalence and p is a fibration.

The fiber of p is the *homotopy fiber* of f, written $F(f)$:

$$F(f) = \{(x, \omega) \in X \times Y_*^I : \omega(1) = f(x)\}.$$

Here we take $0 \in I$ as the basepoint, so ω is a path in Y from $*$ to $f(x)$.

As in our discussion of the Barratt-Puppe cofibration sequence, there is an equivalent way of constructing $F(f)$, by replacing $* \to Y$ with a fibration, namely the path space Y_*^I, and forming the pullback over X:

$$F(f) \longrightarrow P(Y) \quad = \quad Y_*^I$$

$$X \longrightarrow Y$$

Lemma 9.4. *Let $p : E \to B$ be a fibration and $* \in B$. The natural map $p^{-1}(*) \to F(p)$ is a homotopy equivalence.*

Proof. Regard $F(p)$ as the pullback of p along $PB \downarrow B$. The induced map on fibers is a homeomorphism; but PB is contractible so the inclusion of the fiber of $F(p) \downarrow PB$ into $F(p)$ is a homotopy equivalence, by Corollary 5.9. $\square$

Continuing with the analogy with cofibrations, the map $p(f) : F(f) \to X$ is a fibration, with fiber ΩX, and we have the *Barratt-Puppe fibration sequence*

$$Y \xleftarrow{f} X \xleftarrow{p} F(f) \xleftarrow{i} \Omega Y \xleftarrow{\Omega f} \Omega X \xleftarrow{\Omega p} \Omega F(f) \xleftarrow{\Omega i} \cdots$$

that is exact. It gives rise to the long exact homotopy sequence:

Lemma 9.5. *Let $(X, A, *)$ be a pointed pair, and let F denote the homotopy fiber of the inclusion $A \to X$. For each $n \geq 1$ there is a natural isomorphism*

$$\pi_n(X, A) \xrightarrow{\cong} \pi_{n-1}(F, *)$$

such that

$$\pi_n(X, *) \longrightarrow \pi_n(X, A, *)$$

$$\pi_{n-1}(\Omega X, *) \xrightarrow{i} \pi_{n-1}(F, *)$$

with ∂, $\pi_{n-1}(A, *)$, p, and $\cong$ maps as shown

commutes.

Corollary 9.6. *The sequence homotopy long exact sequence of a pair is in fact exact; for $n \geq 2$ the set $\pi_n(X, A, *)$ is a group, abelian for $n \geq 3$; and all the maps between groups in the sequence are homomorphisms.*

Furthermore the bottom sequence makes sense (and is exact) even if $A \to X$ is not a subspace inclusion.

Proof of Lemma 9.5. To begin with, notice that $\pi_1(X, A, *)$ is the set of path components of the space of maps

$$(I, \partial I, J_1) \to (X, A, *) \,.$$

This is the space of paths in X from $*$ to some element of a: that is, it's precisely $F(A \to X)$.
    In fact, for any $n \geq 1$, the space of maps

$$(I^n, \partial I^n, J_n) \to (X, A, *)$$

is precisely $\Omega^{n-1} F(A \to X)$. For example, when $n = 2$, an element in the given space is given by a map $I^2 \to X$ that is the basepoint along the bottom and takes values in A along the top – so a path in $F(A \to X)$ – and also is the basepoint along the left and right edges – so it's a loop in $F(A \to X)$.
    The diagram is easily seen to commute. $\square$

There is another perspective on the homotopy long exact sequence, arising from Lemma 9.4.

Lemma 9.7. *Let $p : E \to B$ be a fibration and $* \in E$. Write $*$ also for the image of $*$ in B, and let F be the fiber over $*$. Then*

$$p_* : \pi_*(E, F, *) \to \pi_*(B, *)$$

is an isomorphism.

Proof. $F(p) \to E$ is a fibration, so by Proposition 5.7 and Lemma 9.4,

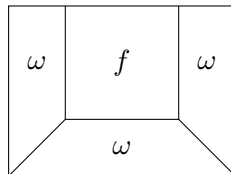$$\mathrm{hofib}(F \to E) \cong \mathrm{hofib}(F(p) \to E) \cong \mathrm{fib}(F(p) \to E) = \Omega B \,.$$

We leave to you the check that the resulting composite isomorphism

$$\pi_n(E, F) \to \pi_{n-1}(\mathrm{hofib}(F \to E)) \to \pi_{n-1}(\Omega B) \to \pi_n(B)$$

is indeed the map induced by p_*. $\square$

We saw that $\pi_1(A, *)$ acts on $\pi_n(A, *)$. The map $\pi_n(A, *) \to \pi_n(X, *)$ is equivariant, if we let $\pi_1(A, *)$ act on $\pi_n(X, *)$ via the group homomorphism $\pi_1(A, *) \to \pi_1(X, *)$. The group $\pi_1(A, *)$ also acts on $\pi_n(X, A, *)$, compatibly.

It's clear from the picture that the maps in the homotopy long exact sequence are equivariant.

Bibliography

[1] Michael Atiyah, Thom complexes, Proceedings of the London Philosophical Society **11** (1961) 291–310.

[2] Michael Atiyah, Bordism and cobordism, Proceedings of the Cambridge Philosophical Society **57** (1961) 200–208.

[3] Aldridge Knight Bousfield, The localization of spaces with respect to homology, Topology **14** (1975) 133–150.

[4] Glen Bredon, *Topology and Geometry*, Graduate Texts in Mathematics **139**, Springer-Verlag, 1993.

[5] Edgar Brown, Cohomology Theories, Annals of Mathematics **75** (1962) 467–484.

[6] Albrecht Dold, *Lectures on Algebraic Topology*, Springer-Verlag, 1995.

[7] Andreas Dress, Zur Spektralsequenz einer Faserung, Inventiones Mathematicae **3** (1967) 172-178.

[8] Björn Dundas, *A Short Course in Differential Topology*, Cambridge Mathematical Textbooks, 2018.

[9] Rudolph Fritsch and Renzo Piccinini, *Cellular Structures in Topology*, Cambridge University Press, 1990.

[10] Martin Frankland, Math 527 - Homotopy Theory: Additional notes, `http://uregina.ca/~franklam/Math527/Math527_0204.pdf`

[11] Paul Goerss and Rick Jardine, *Simplicial Homotopy Theory*, Progress in Mathematics **174**, Springer-Verlag, 1999.

[12] Alan Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[13] Dale Husemoller, *Fiber Bundles*, Graduate Texts in Mathematics **20**, Springer-Verlag, 1993.

[14] Sören Illman, The equivariant triangulation theorem for actions of compact Lie groups. Mathematische Annalen **262** (1983) 487–501.

[15] Niles Johnson, Hopf fibration – fibers and base, `https://www.youtube.com/watch?v=AKotMPGFJYk`.

[16] Dan Kan, Adjoint funtors, Transactions of the American Mathematical Society **87** (1958) 294–329.

[17] Anthony Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics 140, Birkaüser, 2002.

[18] Wolfgang Lück, Survey on classifying spaces for families of subgroups, `https://arxiv. org/ abs/math/0312378`.

[19] Saunders Mac Lane, *Homology*, Springer Verlag, 1967.

[20] Saunders Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5**, Springer-Verlag, 1998.

[21] Jon Peter May, *A Consise Course in Algebraic Topology*, University of Chicago Press, 1999, `https://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`.

[22] Haynes Miller, Leray in Oflag XVIIA: The origins of sheaf theory, sheaf cohomology, and spectral sequences, *Jean Leray (1906–1998)*, Gazette des Mathematiciens **84** suppl (2000) 17– 34. `http://math.mit.edu/~hrm/papers/ss.pdf`

[23] Haynes Miller and Douglas Ravenel, Mark Mahowald's work on the homotopy groups of spheres, *Algebraic Topology, Oaxtepec 1991*, Contemporary Mathematics **146** (1993) 1–30.

[24] John Milnor, The geometric realization of a semi-simplicial complex, Annals of Mathematics **65** (1957) 357–362.

[25] John Milnor, The Steenrod algebra and its dual, Annals of Mathematics **67** (1958) 150–171.

[26] John Milnor, A procedure for killing homotopy groups of differentiable manifolds, Proceedings of Symposia in Pure Mathematics, **III** (1961) 39–55.

[27] John Milnor and Jim Stasheff, *Fiber Bundles*, Annals of Mathematics Studies **76**, 1974.

[28] Steve Mitchell, Notes on principal bundles and classifying spaces.

[29] Steve Mitchell, Notes on Serre Fibrations.

[30] James Munkres, *Topology*, Prentice-Hall, 2000.

[31] Daniel Quillen, *Homotopical Algebra*, Springer Lecture Notes in Mathematics **43**, 1967.

[32] *Hommes de Science: 28 portraits*, Hermann, 1990.

[33] Graeme Segal, Classifying spaces and spectral sequences, Publications mathématiques de l'IHES 34 (1968) 105–112.

[34] Paul Selick, *Introduction to Homotopy Theory*, American Mathematical Society, 1997.

[35] Jean-Pierre Serre, Homologie singulière des espaces fibrés. Applications. Annals of Mathematics **54** (1951), 425–505.

[36] William Singer, Steenrod squares in spectral sequences. I, II. Transactions of the Amererican Mathematical Society **175** (1973) 327–336 and 337–353.

[37] Edwin Spanier, *Algebraic Topology*, McGraw Hill, 1966, and later reprints.

[38] Robert Stong, *Notes on Cobordism Theory*, Mathematical Notes, Princeton University Press, 1968.

[39] Neil Strickland, The category of CGWH spaces, `http://www.neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`.

[40] Arne Strøm, A note on cofibrations, Mathematica Scandinavica **19** (1966) 11–14.

[41] René Thom, Quelques propriétés globales des variétés différentiables, Commentarii Mathematici Helvitici **28** (1954) 17–86.

[42] Tammo tom Dieck, *Algebraic Topology*, European Mathematical Society, 2008.

[43] Kalathoor Varadarajan, *The finiteness obstruction of C. T. C. Wall*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley and Sons, 1989.

[44] Charles Terrence Clegg Wall, Finiteness conditions for CW-complexes, Annals of Mathematics **81** (1965) 56–69.

[45] Charles Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics **38**, 1994.

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