Lectures 21 and 22.

19 The Strong Whitney Embedding Theorem

Whitney proved a stronger version of this theorem.

Theorem 19.1. *(Whitney 1944) Any compact n-manifold admits an embedding into $\mathbb{R}^{2n}$.*

Proof. (Sketch). We will work out the case n is even and $n > 2$ and M orientable first. Consider the space Imm of C^k-immersions of $M \to \mathbb{R}^{2n}$. The condition of being an immersion is an open condition in the C^k-topology on the space of maps so that Imm is a Banach manifold. By Proposition 15.3 proposition this space is non-empty. First we will show that for a Baire set of immersions the there are only finitely many double points and that the two sheets of image are transverse at the double points.

To this end consider the map

$$F : Imm \times (M \times M \setminus \Delta) \to \mathrm{Gr}_n(\mathbb{R}^{2n}) \times \mathrm{Gr}_n(\mathbb{R}^{2n}) \times \mathbb{R}^{2n}.$$

given by $F(f, x, y) = (\mathrm{Im}(D_x f), \mathrm{Im}(D_y f), f(x) - f(x')$. One checks that F is a submersion. Let $Z_i \subset \mathrm{Gr}_n(\mathbb{R}^{2n}) \times \mathrm{Gr}_n(\mathbb{R}^{2n})$ be the set of pairs (Π_1, Π_2) so that $\dim(\Pi_1 \cap \Pi_2) = i$.

Lemma 19.2. *Z_i is a smooth submanifold of dimension $2n^2 - i^2$.*

Proof. Write $\mathbb{R}^{2n}$ as

$$\Pi_1 \cap \Pi_2 \oplus \Pi_1 \cap \Pi_2^\perp \oplus \Pi_1^\perp \cap \Pi_2 \oplus \Pi_1 \cap \Pi_2^\perp$$

The standard coordinate chart about Π_1 represents a plane near Π_1 as the graph of a linear map $A_1 : \Pi_1 \to \Pi_1^\perp$ decomposing this matrix according the to the above deomposition we can write

$$A_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix}$$

viewed as a map

$$\Pi_1 \cap \Pi_2 \oplus \Pi_1 \cap \Pi_2^\perp \to \Pi_1^\perp \cap \Pi_2^\perp \oplus \Pi_1^\perp \cap \Pi_2$$

Doing the same of a chart about Π_2 we get

$$A_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{bmatrix}$$

now viewed as a map

$$\Pi_1 \cap \Pi_2 \oplus \Pi_1^\perp \cap \Pi_2 \to \Pi_1^\perp \cap \Pi_2^\perp \oplus \Pi_1 \cap \Pi_2^\perp$$

The condition that the planes represented by (A_1, A_2) also intersect in an i-dimensional subspace is the condition that $\alpha_1 = \alpha_2$ so the total dimension is $2n^2 - i^2$ $\square$

We seek a map f so that for all distinct $x, y \in M$ $F(f, x, y) \notin Z_i \times \{0\}$ for any i. The parametric transversality theorem implies that for a Baire set of f the map $(x, y) \mapsto F(f, x, y)$ is transverse to $Z_i \times \{0\}$. But the codimension of $Z_i \times \{0\}$ is $2n^2 + 2n - (2n^2 - i^2) = i^2 + 2n$ which is larger than the dimension of the domain $2n$.

Exercise 8. Show that we can in addition assume that f has no triple points.

Thus whenever $f(x) = f(y)$ we have that the differentials have transverse images at those points. We assume that in the remainder of the discussion that f has been chosen satisfy these conditions.

Lemma 19.3. *At each pair (x, x') with $f(x) = f(x') = y$ there are charts (U, ϕ), (U', ϕ') near x, x' and (V, ψ) near y so that*

$$\psi^{-1} \circ f \circ \phi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0)$$

and

$$\psi^{-1} \circ f \circ \phi'(x_1', x_2', \ldots, x_n') = (0, 0, \ldots, 0, x_1', x_2', \ldots, x_n')$$

Proof. Since f is an immersion there are coordinates $\phi = (x_1, \ldots, x_n)$ about x and $\psi_1(y_1, \ldots, y_{2n})$ about y so that

$$\psi_1^{-1} \circ f \circ \phi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0)$$

and coordinates $\phi' = (x_1', \ldots, x_n')$ about x' and $\psi_2 = (y_1', \ldots, y_{2n}')$ about y so that

$$\psi_2^{-1} \circ f \circ \phi(x_1', x_2', \ldots, x_n') = (0, 0, \ldots, 0, x_1', x_2', \ldots, x_n')$$

Then set $\psi = (y_1, \ldots, y_n, y_{n+1}', \ldots, y_{2n}')$ We claim that this gives the desired coordinate system. $\square$

Thus the double points are isolated and hence by compactness there are finitely many.

Next we define the sign of a double point. Recall now that are assuming that n is even and that M is orientable. Choose an orientation of M and of $\mathbb{R}^{2n}$. If $f(x) = f(x') = y$ then transversality tells us that we can write

$$T_y\mathbb{R}^{2n} = D_x f(T_x M) \oplus D_{x'} f(T_{x'} M).$$

As both sides of this equations are oriented vector spaces we can assign a sign to the double point according to whether or not the orientations agree. Notice that since n is even the order of the factors on the right hand side is immaterial. Also notice that the sign is independent of the choice of orientation of M.

We will now prove the following key proposition.

Proposition 19.4. *If a pair of double points y_1 and y_2 of opposite sign with preimages (x_1, x_1') and (x_2, x_2') respectively. Then we can modify f so as to eliminate the double point without introducing any others.*

Proof. Then choose γ and γ' embedded smooth curves in M with endpoints x_1, x_2 and x_1', x_2' respectively. Since $n > 2$ we can assume that the curves are disjoint and that their images are disjoint except at the endpoints. Let $\Gamma = f(\gamma) \cup f(\gamma')$ denote the union of these images. Γ is an embedded closed curve in $\mathbb{R}^{2n}$ and hence bounds a disk $\sigma : D^2 \to \mathbb{R}^{2n}$. We can assume that σ is transverse to f and to itself. This implies that σ has no double points and that σ misses f except along Γ.

Let N be the normal bundle of σ. Since σ is contractible N is trivial so that there is a bundle isomorphism

$$N \equiv D^2 \times \mathbb{R}^{2n-2}.$$

Let ν and ν' denote the normal bundles of γ and γ' in M. These are again trivial bundles. Note that along $f(\gamma)$, $Df(\nu)$ defines a distinguished subbundle similarly along $f(\gamma')$, $Df(\nu')$.

Notice that the tubular neighborhood of By the tubular neighborhood theorem there is a diffeomorphism

$$\psi : D^2 \times D^{2n-2} \to \mathbb{R}^{2n}$$

Suppose that we can write $N = \xi_1 \oplus \xi_2$ so that

$$\xi|_{f(\gamma)} = Df(\nu) \quad \text{and} \quad \xi|_{f(\gamma')} = Df(\nu')$$

Then we can write the tubular neighborhood of σ in a standard way and we see since we can push the two dimensional picture till the two arcs don't intersect we can also push the higher dimensional picture till they don't intersect.

We must return to the issue of extending the splitting. The splitting gives rise to a map $\upsilon : \Gamma \to \mathrm{Gr}_{n-1}(\mathbb{R}^{2n-2})$ and we must understand when this map is null homotopic. Form algebraic topology we know that $\mathrm{Gr}_{n-1}(\mathbb{R}^{2n-2})$ fundamental group $\mathbb{Z}/2\mathbb{Z}$ and is generated by the family of subspaces

$$\Pi_t = \mathrm{span}\{\cos(t)e^1 + \sin(t)e^n, e^2, \ldots, e^{n-1}\}.$$

as t varies between 0 and π. In other words the identification of Π_0 with Π_π is orientation reversing. Thus the orientation of $\xi_1 \oplus \xi_2$ must be the same at the two end if the splitting is to extend. On the other hand the normal vectors in the two disk reverse orientation.

$\square$

To prove the theorem we need to see that we first modify f so that the signed number of double points is zero. To this end consider the map

$$(x_1, \ldots, x_n) \mapsto (x_1 - 2x_1/u, x_2, \ldots, x_n, 1/u, x_1x_2/u, \ldots, x_1x_n/u)$$

where $u(x_1, \ldots, x_n) = (1+x_1^2)(1+x_2^2)\ldots(1+x_n^2)$. It is straightfoward if tedious to check that this map has exactly one double point and also notice that at very large distance from the origin this map is quite close to the linear embedding

$$(x_1, \ldots, x_n, 0, \ldots, 0)$$

in other words we can shrink the map down a lot and use it to modify a given map to have another double point and we can choose the sign of this double points as well.

Now we consider the case that n is odd (it doesn't matter now if M is orientable). Then the sign of a point of intersection is not well defined. In this case however the relative sign of a pair of intersection points given the pair of curves γ and γ' is still well defined. If the curves γ joining x_1 and x_2 and γ' joining x_1' to x_2' lead to intersection with the same sign choose different curves now joining x_1 to x_2' and joining x_1' to x_2.

If M is nonorientable and we have curves γ and γ' leading to a pair of intersection point with the same sign add to γ' a curve running around a loop that reverses orientation. $\square$