SYMPLECTIC GEOMETRY, LECTURE 1

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1. Differential forms

Given M a smooth manifold, one has two natural bundles: the *tangent bundle* $TM = \{v = \sum v_i \frac{\partial}{\partial x_i}\}$ and the *cotangent bundle* $T^*M = \{\alpha = \sum \alpha_i dx_i\}$. Under C^∞ maps, tangent vectors pushforward:

$$f : M \to N \implies f_*(v) = df(v) \in T_{f(v)}N \tag{1}$$

Similarly, differential forms pull back: $f^*(\alpha) = \alpha \circ df \in T_p^*M$.

Definition 1. *A* differential p-form *is a section of* $\bigwedge^p T^*M$. *We denote the set of such sections as*

$$\Omega^p(M) = \Omega^p(M, \mathbb{R}) = C^\infty(\bigwedge^p T^*M) \tag{2}$$

Recall that, for E a vector space, $\bigwedge^* E = \bigotimes^* E / \{e_i \wedge e_j + e_j \wedge e_i = 0\}$. Furthermore, $\bigwedge^* E$ has a basis $e_{i_1} \wedge \cdots \wedge e_{i_p}$, $i_1 < \cdots < i_p$. In coordinates, a p-form is locally

$$\alpha = \sum_{i_1 < \cdots < i_p} \alpha_{i_1, \cdots, i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \tag{3}$$

where the $\alpha_{i_1, \cdots, i_p}$ are C^∞ functions. (Under coordinate changes, $x_i = f_i(y_1, \ldots, y_n)$, one replaces dx_i by $df_i = \sum_j \frac{\partial f_i}{\partial y_j} dy_j$.)

Definition 2. *The* exterior differential *is the map* $d : \Omega^p \to \Omega^{p+1}$ *which maps:*

- *For f a function,* $df = \sum \frac{\partial f}{\partial x_i} dx_i$.
- $d(f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

d is obtained by extending $\mathbb{R}$-linearly to all of Ω^p.

Note that d satisfies $d(f\alpha) = f d\alpha + df \wedge \alpha$. The exterior derivative has the following properties:

- $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$. In coordinates,

$$d((f dx_{i_1} \wedge \cdots \wedge dx_{i_p}) \wedge (g dx_{j_1} \wedge \cdots \wedge dx_{j_q})) = (f dg + g df) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_q} \tag{4}$$

- $d^2 = 0$. For any function f,

$$d^2 f = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_i \wedge dx_j = 0 \tag{5}$$

  because terms with switched i, j cancel.

These two properties give us the structure of a differential graded algebra on $\Omega^*(M) = \bigoplus_p \Omega^p(M)$.

- $\forall \phi \in C^\infty(M, N), \alpha \in \Omega^p(N), \phi^*(d\alpha) = d(\phi^*\alpha)$.

Other operations:

- For $v \in C^\infty(TM)$ a vector field, $\alpha \in \Omega^p(M)$ a form, we have the *interior product* $i_v\alpha = \alpha(v, \cdots) \in \Omega^{p-1}(M)$.
- For $X \in C^\infty(TM)$ a vector field, $f \in C^\infty(M)$, we have the *Lie derivative* $X \cdot f = L_X f = i_X df = df(X)$. If X generates diffeomorphisms ϕ^t on M with $\phi^0(x) = x$ and $\frac{d}{dt}\phi^t(x) = X(\phi^t(x))$, then

$$\frac{d}{dt}((\phi^t)^* f) = \frac{d}{dt}(f \circ \phi^t) = \phi^{t*}(X \cdot f) \tag{6}$$

We can extend this construction to forms: given $\alpha \in \Omega^p(M), X \in C^\infty(TM)$ a vector field, $L_X\alpha \in \Omega^p$ is defined s.t.

$$\frac{d}{dt}((\phi^t)^*\alpha) = \phi_t^*(L_X\alpha) \tag{7}$$

Note that the Lie derivative satisfies

$$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta \tag{8}$$

and $L_X(d\alpha) = d(L_X\alpha)$.

Combining these two properties, we find that:

Proposition 1. *$L_X\alpha = di_X\alpha + i_Xd\alpha$.*

Proof. By induction: base case is trivial, so assume statement for p-forms. Locally, a $(p+1)$ form is the sum of $fd\alpha$ for $f \in C^\infty(M), \alpha \in \Omega^p$. Thus,

$$\begin{aligned} L_X(fd\alpha) &= (L_Xf)d\alpha + fdL_X\alpha \\ &= (i_Xdf)d\alpha + f(ddi_X\alpha + di_Xd\alpha) \\ &= (i_Xdf)d\alpha + fdi_Xd\alpha \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} di_X(fd\alpha) + i_Xd(fd\alpha) &= d(fi_Xd\alpha) + i_X(df \wedge d\alpha) \\ &= df \wedge i_Xd\alpha + fdi_Xd\alpha + (i_Xdf)d\alpha - df \wedge i_Xd\alpha \\ &= (i_Xdf)d\alpha + fdi_Xd\alpha \end{aligned} \tag{10}$$

giving us the desired equality. □

2. de Rham cohomology

Definition 3. *We say that $\alpha \in \Omega^p$ is* closed *if $d\alpha = 0$,* exact *if $\alpha = d\beta$ for some β. The* de Rham cohomology *of M is the collection of groups*

$$H^p(M, \mathbb{R}) = \frac{ker(d : \Omega^p \to \Omega^{p+1})}{\text{Im }(d : \Omega^{p-1} \to \Omega^p)} \tag{11}$$

Example. For M connected, $df = 0 \Leftrightarrow f$ is constant, so $H^0(M, \mathbb{R}) = \mathbb{R}$.

Proposition 2 (Poincaré Lemma)**.** *$H^p(\mathbb{R}^n) = 0\ \forall p \geq 1$.*

Proof. By induction on n. The case $n = 1$ is obvious, as $f = \int \alpha dx \implies df = \alpha$. For general n, write

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \alpha_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \tag{12}$$

on $\mathbb{R}^n$ and assume α is closed. Let

$$\beta = \sum_{2 \leq j_1 < \cdots < j_{p-1} \leq n} \beta_{j_1 \cdots j_{p-1}} dx_{j_1} \wedge \cdots \wedge dx_{j_{p-1}} \tag{13}$$

where $\frac{\partial \beta_{j_1 \cdots j_{p-1}}}{\partial x_1} = \alpha_{1 j_1 \cdots j_{p-1}}$ (i.e. $\beta_{j_1 \cdots j_p} = \int \alpha_{1 j_1 \cdots j_{p-1}} dx_1$). Then $i_{\frac{\partial}{\partial x_1}} d\beta = i_{\frac{\partial}{\partial x_1}} \alpha$ by construction. Let $\alpha' = \alpha - d\beta$. Then $\alpha' = \sum_{2 \leq i_1 < \cdots < i_p \leq n} \alpha'_{i_1 \cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ with no dx_1 by construction and $d\alpha' = d\alpha - d(d\beta) = 0$, showing that α' is pulled back from $\mathbb{R}^{n-1}$ by $(x_1, \ldots, x_n) \overset{\pi}{\mapsto} (x_2, \ldots, x_n)$. Writing $\alpha' = \pi^*\eta, \eta \in \Omega(\mathbb{R}^{n-1})$, we have that $d\eta = 0$ and $\eta = d\gamma$ by our inductive hypothesis. Thus, $\alpha = \alpha' + d\beta = d(\pi^*\gamma + \beta)$ as desired. □

2.1. Variants of de Rham Cohomology.

- If M is noncompact, we can also consider the space of *compactly supported differential forms* $\Omega^p_c(M, \mathbb{R})$ and get the associated compactly supported de Rham cohomology $H^p_c(M, \mathbb{R})$.
- If $U \subset M$ is a submanifold (e.g. an open subset), we can define relative differential forms $\Omega^p(M, U; \mathbb{R}) = \{\alpha \in \Omega^p(M, \mathbb{R}) | \alpha|_U = 0\}$ and obtain the *relative de Rham cohomology* $H^p(M, U; \mathbb{R})$.

3. Exact sequences of complexes

If $M = U \cup V$, $U, V \subset M$ open, we have an exact sequence on forms

$$0 \to \Omega^p(M) \to \Omega^P(U) \oplus \Omega^p(V) \to \Omega^p(U \cap V) \to 0 \tag{14}$$

where the first map sends $\alpha \mapsto (\alpha|_U, \alpha|_V)$ and the second $(\alpha, \beta) \to \alpha|_{U\cap V} - \beta|_{U\cap V}$. Both these maps commute with d, and exactness is clear: for the surjectivity of the last map, use a partition of unity $1 = u + v$, where $\text{supp}(u) \subset U, \text{supp}(v) \subset V$, so $\gamma \in \Omega^p(U \cap V)$ is the image of $(v\gamma, -u\gamma)$. This short exact sequence then gives a long exact sequence (called the *Mayer-Vietoris* sequence)

$$\cdots \to H^p(M) \to H^p(U) \oplus H^p(V) \to H^p(U \cap V) \xrightarrow{\delta} H^{p+1}(M) \to \cdots \tag{15}$$

The map δ is obtained as follows:

1. Choose a splitting $\sigma : \Omega^p(U \cap V) \to \Omega^p(U) \oplus \Omega^p(V)$.
2. Given $\gamma \in \Omega^p(U \cap V)$ closed, $d\sigma(\gamma)$ lands in the image of $i^* : \Omega^{p+1}(M) \to \Omega^{p+1}(U) \oplus \Omega^{p+1}(V)$, and its preimage gives the desired element of $\Omega^{p+1}(M)$.

Similarly, for $U \subset M$, we get a sequence $0 \to \Omega^p(M, U) \to \Omega^p(M) \to \Omega^p(U) \to 0$, with the maps given by inclusion and restriction respectively, and thus a long exact sequence of relative cohomology. Using these properties along with Poincaré duality and functoriality under diffeomorphisms, we get

Theorem 1. *The de Rham and singular (simplicial) cohomologies are equivalent.*

3.1. Operations on de Rham cohomology.

- Cup product: $[\alpha] \cup [\beta] = [\alpha \wedge \beta]$. This is well defined: $d\alpha = d\beta = 0 \implies d(\alpha \wedge \beta) = 0$, and $(\alpha + d\eta) \wedge \beta = \alpha \wedge \beta + d(\eta \wedge \beta)$.
- Pairing with homology: for $\Sigma \subset M$ a p-dimensional submanifold which is oriented and closed, we have an element $[\Sigma] \in H_p(M)$ and thus a pairing $\langle [\alpha], [\Sigma] \rangle = \int_\Sigma \alpha$. More generally, given a p-cycle $[\Sigma]$ represented by $\sum n_i C_i$, with C_i p-dimensional submanifolds with ∂, we get the same pairing extended linearly. That this is well-defined is a consequence of Stokes' theorem $\int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha$.
- Poincaré duality: For M^n compact, $[\alpha] \in H^p(M), [\beta] \in H^{n-p}(M) \mapsto \int_M \alpha \wedge \beta = ([\alpha] \cup [\beta]) \cdot [M]$ is a nondegenerate linear pairing and gives an isomorphism $H^{n-p} \cong H_p$. In the noncompact case, we have $[\alpha] \in H^p(M), [\beta] \in H_c^{n-p}(M) \mapsto \int_M \alpha \wedge \beta$ giving $H_c^{n-p} \cong H_p$.