18.969 Topics in Geometry: Mirror Symmetry
Spring 2009

MIRROR SYMMETRY: LECTURE 20

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1. Homological Mirror Symmetry (cntd.)

Last time, we studied homological mirror symmetry on T^2 (with area form λ) on the one hand and $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}, \tau = i\lambda$ on the other. Lagrangians of slope (p, q) with a $U(1)$ flat connection correspond to vector bundles of rank p and degree $-q$ (for $(p, q) = (0, -1)$, this gives skyscraper sheaves). We showed that m_2 corresponds to theta functions and to sections and products.

1.1. **Massey Products.** We consider these in the special case of a triangulated category $\mathcal{D}$, and consider objects and morphisms $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \xrightarrow{h} X_4$ where $g \circ f = 0, h \circ g = 0$. Assume that $\hom(X_1, X_3[-1]) = \hom(X_2, X_4[-1]) = 0$. Then $m_3(h, g, f) \in \hom(X_1, X_4[-1])$. Let K be s.t. $K \to X_2 \xrightarrow{g} X_3 \xrightarrow{[1]} K[1]$ is a distinguished triangle (i.e. $K[1] = \mathrm{Cone}(g)$). Then $g \circ f = 0 \implies f$ factors through $X_1 \xrightarrow{\overline{f}} K \to X_2$, where $\overline{f} \in \hom(X_1, K)$ comes from

$$\hom(X_1, X_3[-1]) \to \hom(X_1, K) \to \hom(X_1, X_2) \xrightarrow{g} \hom(X_1, X_3) \tag{1}$$

Similarly, $h \circ g = 0 \implies h$ factors through $X_3 \to K[1] \xrightarrow{\overline{h}} X_4$, and we define

$$m_3(h, g, f) := \overline{h}[-1] \circ \overline{f} : X_1 \xrightarrow{\overline{f}} K \xrightarrow{\overline{h}[-1]} X_4[-1] \tag{2}$$

Now, let's say that we had f, g, h in the A_∞ category of twisted complexes, $K = \{X_2 \xrightarrow{g} X_3[-1]\}$,

$$\begin{tikzcd} X_1 \arrow[d, "f"] & \\ X_2 \arrow[r, "g"] & X_3[-1] \arrow[d, "h[-1]"] \\ & X_4[-1] \end{tikzcd} \tag{3}$$

and $m_2^{\mathrm{Tw}}(\overline{h}[-1], \overline{f}) = m_3(h, g, f)$. If we add an extra step

$$\begin{tikzcd}
e = & X_2 \arrow[r, "g"] \arrow[d, "\mathrm{id}"'] & X_3[-1] \arrow[d, "0"] & , \; m_1(e) = & X_1 \arrow[d, "f"] & \\
& X_2 \arrow[r, "g"'] & X_3[-1] & & X_2 \arrow[r] \arrow[dr, "g"] & X_3[-1] \\
& & & & X_2 \arrow[r] & X_3[-1] \arrow[d, "h"] \\
& & & & & X_4[-1]
\end{tikzcd} \tag{4}$$

then we get

$$m_3(h, g, f) = m_3(h, m_1(e), f) = m_2(h, m_2(e, f)) + \text{ other terms which vanish} \tag{5}$$

Now, let $\mathcal{L} \to X^\vee$ be a nontrivial degree 0 holomorphic line bundle over an elliptic curve, p, q generic points. Then the pairwise compositions in

$$\mathcal{O} \xrightarrow{f} \mathcal{O}_p \xrightarrow{g} \mathcal{L}[1] \xrightarrow{h} \mathcal{O}_q[1] \tag{6}$$

vanish, and we have

$$\begin{aligned} \hom(\mathcal{O}_p, \mathcal{L}[1]) &= \mathrm{Ext}^1(\mathcal{O}_p, \mathcal{L}) \cong \mathrm{Hom}(\mathcal{L}, \mathcal{O}_p)^\vee \\ \hom(\mathcal{O}, \mathcal{L}[1]) &= \mathrm{Ext}^1(\mathcal{O}, \mathcal{L}) \cong H^1(\mathcal{L}) = 0 \end{aligned} \tag{7}$$

Then $K \cong \mathcal{L} \otimes \mathcal{O}(p)$ is a degree 1 line bundle, neither $\mathcal{O}(p)$ nor $\mathcal{O}(q)$: note that $\mathcal{O}(p)$ is a degree 1 line bundle with a section $s_p, s_p^{-1}(0) = \{p\}$. Then we have a long exact sequence

$$0 \to \mathcal{L} \xrightarrow{s_p} \mathcal{L} \otimes \mathcal{O}(p) \to \mathcal{O}_p \to 0 \tag{8}$$

giving us an exact triangle in the derived category

$$K = \mathcal{L} \otimes \mathcal{O}(p) \to \mathcal{O}_p \xrightarrow{g} \mathcal{L}[1] \xrightarrow{[1]} K[1] \tag{9}$$

via our extension class. f should factor as a map from $\mathcal{O}$ to K, and does via a nontrivial section $\overline{f}$ of $K = \mathcal{L} \otimes \mathcal{O}(p)$. Moreover, for $\overline{h}[-1]$ nontrivial in $\hom(K, \mathcal{O}_q)$, $\overline{h}[-1] \circ \overline{f} \neq 0$.

This matches with the calculation of m_3 for the relevant Lagrangians in the Fukaya category of T^2: two horizontal lines and two vertical lines, bounding an infinite series of rectangles. See notes for a visual description of this.

2. Strominger-Yau-Zaslow (SYZ) Conjecture

Motivating question: how does one build a mirror $X^\vee$ of a given Calabi-Yau X? Observe that homological mirror symmetry (1994) says that $D^b\mathrm{Coh}(X^\vee) \cong D^\pi\mathrm{Fuk}(X)$. Points $p \in X^\vee$ correspond to skyscraper sheaves $\mathcal{O}_p \in D^b\mathrm{Coh}(X^\vee)$ and $\mathcal{L}_p \in D^\pi\mathrm{Fuk}(X)$. That is, we can regard $X^\vee$ as the moduli space of skyscraper sheaves in $D^b\mathrm{Coh}(X^\vee)$ as well as a moduli space of certain objects of $D^\pi\mathrm{Fuk}(X)$. The question reduces to understanding exactly what are these certain objects. Four lectures ago, we computed $\mathrm{Ext}^k(\mathcal{O}_p, \mathcal{O}_p) \cong \bigwedge^k V$ for V the tangent space at p. As a graded vector space, $\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong H^*(T^n; \mathbb{C})$. Four lectures before that, we showd that $HF^*(L, L)$ is in good cases isomorphic to $H^*(L)$, but if L bounds disks, these are only related by a spectral sequence.

Remark. Warning: recall that in general we are dealing with Λ-coefficients. In good cases, we can set $T = e^{-2\pi}$ and hope that we have convergence. If convergence fails, we only get a formal family near LSCL.

If (optimistically) we assume $\mathcal{L}_p$ is an actual Lagrangian, then it should be a Lagrangian torus. There are not enough of these: given $T^n \cong L \subset X, V(L) \cong T^*L$, one has that Lagrangian deformations of L are graphs of closed 1-forms, while Hamiltonian isotopies are graphs of exact 1-forms. Furthermore, for T^n, $\mathrm{Def}_L \cong H^1(L, \mathbb{R}) \simeq \mathbb{R}^n$.

Now, recall the twisted Floer homology for pairs (L, ∇), with ∇ a flat $U(1)$ connection on $\underline{\mathbb{C}} \to L$: $\nabla = d + A, A \in \Omega^1(L, i\mathbb{R})$. Taking this modulo gauge tranformations and exact 1-forms, we obtain $H^1(L; i\mathbb{R})$. One can hope that generic points of $X^\vee$ parameterize isomorphism classes of (L, ∇), $L \subset X$ a Lagrangian torus and $U(1)$ a flat connection.