Chapter 2

A Review on Differentiation

Reading: Spivak pp. 15-34, or Rudin 211-220

2.1 Differentiation

Recall from 18.01 that

Definition 2.1.1. *A function* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is* **differentiable** *at* $a \in \mathbb{R}^n$ *if there exists a linear transformation* $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ *such that*

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0 \tag{2.1}$$

The norm in Equation 2.1 is essential since $f(a+h) - f(a) - \lambda(h)$ is in $\mathbb{R}^m$ and h is in $\mathbb{R}^n$.

Theorem 2.1.2. *If* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is differentiable at* $a \in \mathbb{R}^n$*, then there is a* **unique** *linear transformation* $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ *that satisfies Equation* (2.1)*. We denote* λ *to be* $Df(a)$ *and call it the* **derivative** *of* f *at* a

Proof. Let $\mu : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \mu(h)|}{|h|} = 0 \tag{2.2}$$

and $d(h) = f(a+h) - f(a)$, then

$$\lim_{h \to 0} \frac{|\lambda(h) - \mu(h)|}{|h|} = \lim_{h \to 0} \frac{|\lambda(h) - d(h) + d(h) - \mu(h)|}{|h|} \tag{2.3}$$

$$\leq \lim_{h \to 0} \frac{|\lambda(h) - d(h)|}{|h|} + \lim_{h \to 0} \frac{|d(h) - \mu(h)|}{|h|} \tag{2.4}$$

$$= 0. \tag{2.5}$$

Now let $h = tx$ where $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, then as $t \to 0$, $tx \to 0$. Thus, for $x \neq 0$, we have

$$\lim_{t \to 0} \frac{|\lambda(tx) - \mu(tx)|}{|tx|} = \frac{|\lambda(x) - \mu(x)|}{|x|} \tag{2.6}$$

$$= 0 \tag{2.7}$$

Thus $\mu(x) = \lambda(x)$. □

Although we proved in Theorem 2.1.2 that if $Df(a)$ exists, then it is unique. However, we still have not discovered a way to find it. All we can do at this moment is just by guessing, which will be illustrated in Example 1.

Example 1. *Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by*

$$g(x, y) = \ln x \tag{2.8}$$

Proposition 2.1.3. *$Dg(a, b) = \lambda$ where λ satisfies*

$$\lambda(x, y) = \frac{1}{a} \cdot x \tag{2.9}$$

Proof.

$$\lim_{(h,k) \to 0} \frac{|g(a+h, b+k) - g(a,b) - \lambda(h,k)|}{|(h,k)|} = \lim_{(h,k) \to 0} \frac{|\ln(a+h) - \ln(a) - \frac{1}{a} \cdot h|}{|(h,k)|} \tag{2.10}$$

Since $\ln'(a) = \frac{1}{a}$, we have

$$\lim_{h \to 0} \frac{|\ln(a+h) - \ln(a) - \frac{1}{a} \cdot h|}{|h|} = 0 \tag{2.11}$$

Since $|(h,k)| \geq |h|$, we have

$$\lim_{(h,k) \to 0} \frac{|\ln(a+h) - \ln(a) - \frac{1}{a} \cdot h|}{|(h,k)|} = 0 \tag{2.12}$$

□

Definition 2.1.4. *The* **Jacobian matrix** *of f at a is the $m \times n$ matrix of $Df(a) : \mathbb{R}^n \to \mathbb{R}^m$ with respect to the usual bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, and denoted $f'(a)$.*

Example 2. *Let g be the same as in Example 1, then*

$$g'(a,b) = (\frac{1}{a}, 0) \tag{2.13}$$

Definition 2.1.5. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is* **differentiable on** *$A \subset \mathbb{R}^n$ if f is diffrentiable at a for all $a \in A$. On the other hand, if $f : A \to \mathbb{R}^m, A \subset \mathbb{R}^n$, then f is called* **differentiable** *if f can be extended to a differentiable function on some open set containing A.*

2.2 Properties of Derivatives

Theorem 2.2.1. *1. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a constant function, then $\forall a \in \mathbb{R}^n$,*

$$Df(a) = 0. \tag{2.14}$$

2. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $\forall a \in \mathbb{R}^n$

$$Df(a) = f. \tag{2.15}$$

Proof. The proofs are left to the readers □

Theorem 2.2.2. *If $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by $g(x, y) = xy$, then*

$$Dg(a, b)(x, y) = bx + ay \tag{2.16}$$

In other words, $g'(a, b) = (b, a)$

Proof. Substitute p and Dp into L.H.S. of Equation 2.1, we have

$$\begin{aligned}
\lim_{(h,k)\to 0} \frac{|g(a+h, b+k) - g(a, b) - Dg(a, b)(h, k)|}{|(h, k)|} &= \lim_{(h,k)\to 0} \frac{|hk|}{|(h, k)|} && (2.17)\\
&\leq \lim_{(h,k)\to 0} \frac{\max(|h|^2, |k|^2)}{\sqrt{h^2+k^2}} && (2.18)\\
&\leq \sqrt{h^2+k^2} && (2.19)\\
&= 0 && (2.20)
\end{aligned}$$

□

Theorem 2.2.3. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, and $g : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $f(a)$, then the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at a, and*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a) \tag{2.21}$$

Proof. Put $b = f(a)$, $\lambda = f'(a)$, $\mu = g'(b)$, and

$$u(h) = f(a+h) - f(a) - \lambda(h) \tag{2.22}$$

$$v(k) = g(b+k) - g(b) - \mu(k) \tag{2.23}$$

for all $h \in \mathbb{R}^n$ and $k \in \mathbb{R}^m$. Then we have

$$|u(h)| = \epsilon(h)|h| \tag{2.24}$$

$$|v(k)| = \eta(k)|k| \tag{2.25}$$

where

$$\lim_{h \to 0} \epsilon(h) = 0 \tag{2.26}$$
$$\lim_{k \to 0} \eta(k) = 0 \tag{2.27}$$

Given h, we can put k such that $k = f(a+h) - f(a)$. Then we have

$$|k| = |\lambda(h) + u(h)| \leq [\|\lambda\| + \epsilon(h)]|h| \tag{2.28}$$

Thus,

$$\begin{aligned} g \circ f(a+h) - g \circ f(a) - \mu(\lambda(h)) &= g(b+k) - g(b) - \mu(\lambda(h)) && (2.29) \\ &= \mu(k - \lambda(h)) + v(k) && (2.30) \\ &= \mu(u(h)) + v(k) && (2.31) \end{aligned}$$

Thus

$$\frac{|g \circ f(a+h) - g \circ f(a) - \mu(\lambda(h))|}{|h|} \leq \|\mu\|\epsilon(h) + [\|\lambda\| + \epsilon(h)]\eta(h) \tag{2.32}$$

which equals 0 according to Equation 2.26 and 2.27. □

Exercise 1. *(Spivak 2-8) Let $f : \mathbb{R} \to \mathbb{R}^2$. Prove that f is differentiable at $a \in \mathbb{R}$ if and only if f^1 and f^2 are, and that in this case*

$$f'(a) = \begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \end{pmatrix} \tag{2.33}$$

Corollary 2.2.4. *If $f : \mathbb{R}^n \to \mathbb{R}^m$, then f is differentiable at $a \in \mathbb{R}^n$ if and* *only if each f^i is, and*

$$\lambda'(a) = \begin{pmatrix} (f^1)'(a) \\ (f^2)'(a) \\ . \\ . \\ . \\ (f^m)'(a) \end{pmatrix}. \tag{2.34}$$

Thus, $f'(a)$ is the $m \times n$ matrix whose ith row is $(f^i)'(a)$

Corollary 2.2.5. *If $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at a, then*

$$D(f+g)(a) = Df(a) + Dg(a) \tag{2.35}$$

$$D(fg)(a) = g(a)Df(a) + f(a)Dg(a) \tag{2.36}$$

$$D(f/g)(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{[g(a)]^2}, \; g(a) \neq 0 \tag{2.37}$$

Proof. The proofs are left to the readers. □

2.3 Partial Derivatives

Definition 2.3.1. *If $f : \mathbb{R}^n \to \mathbb{R}$ and $a \in \mathbb{R}^n$, then the limit*

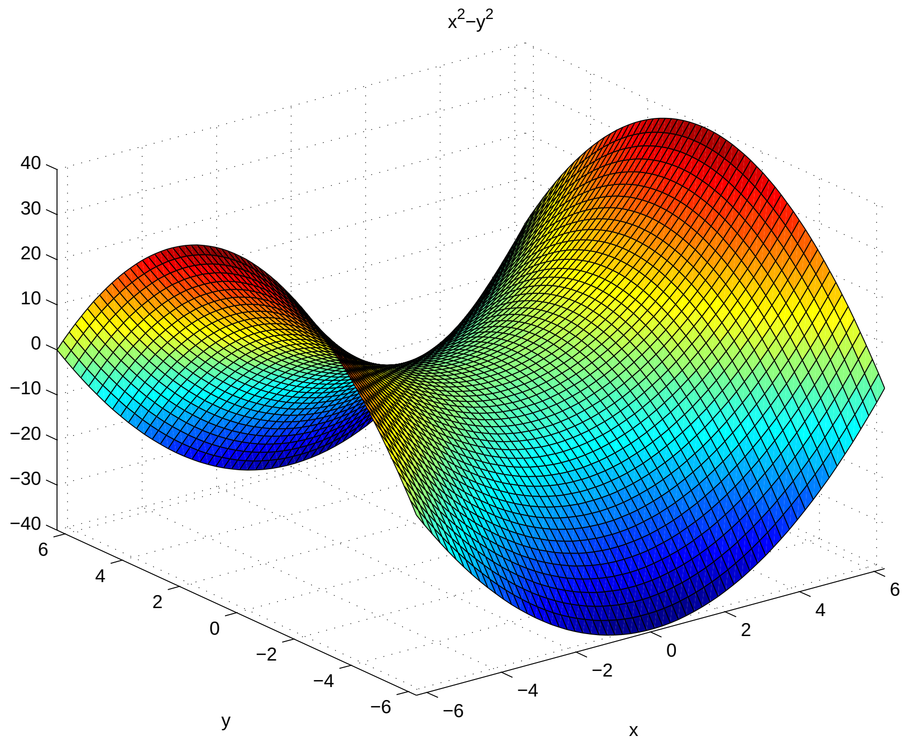
$$D_i f(a) = \lim_{h \to 0} \frac{f(a^1, ..., a^i + h, ..., a^n) - f(a^1, ..., a^n)}{h} \tag{2.38}$$

is called the ith **partial derivative** *of f at a if the limit exists.*

If we denote $D_j(D_i f)(x)$ to be $D_{i,j}(x)$, then we have the following theorem which is stated without proof. (The proof can be found in Problem 3-28 of Spivak)

Theorem 2.3.2. *If $D_{i,j}f$ and $D_{j,i}f$ are continuous in an open set containing a, then*

$$D_{i,j}f(a) = D_{j,i}f(a) \tag{2.39}$$



Partial derivatives are useful in finding the extrema of functions.

Theorem 2.3.3. *Let $A \subset \mathbb{R}^n$. If the maximum (or minimum) of $f : A \to \mathbb{R}$ occurs at a point a in the interior of A and $D_i f(a)$ exists, then $D_i f(a) = 0$.*

Proof. The proof is left to the readers. □

However the converse of Theorem 2.3.3 may not be true in all cases. (Consider $f(x, y) = x^2 - y^2$).

2.4 Derivatives

Theorem 2.4.1. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then $D_j f^i(a)$ exists for $1 \leq i \leq m, 1 \leq j \leq n$ and $f'(a)$ is the $m \times n$ matrix $(D_j f^i(a))$.*