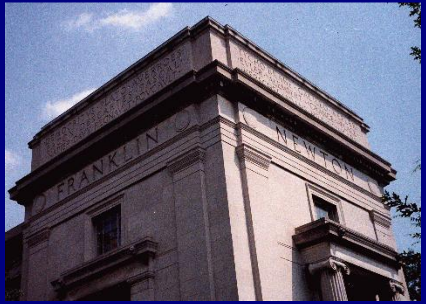
Why are random matrix eigenvalues cool?



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MAA Mathfest 2002
Thursday, August 1

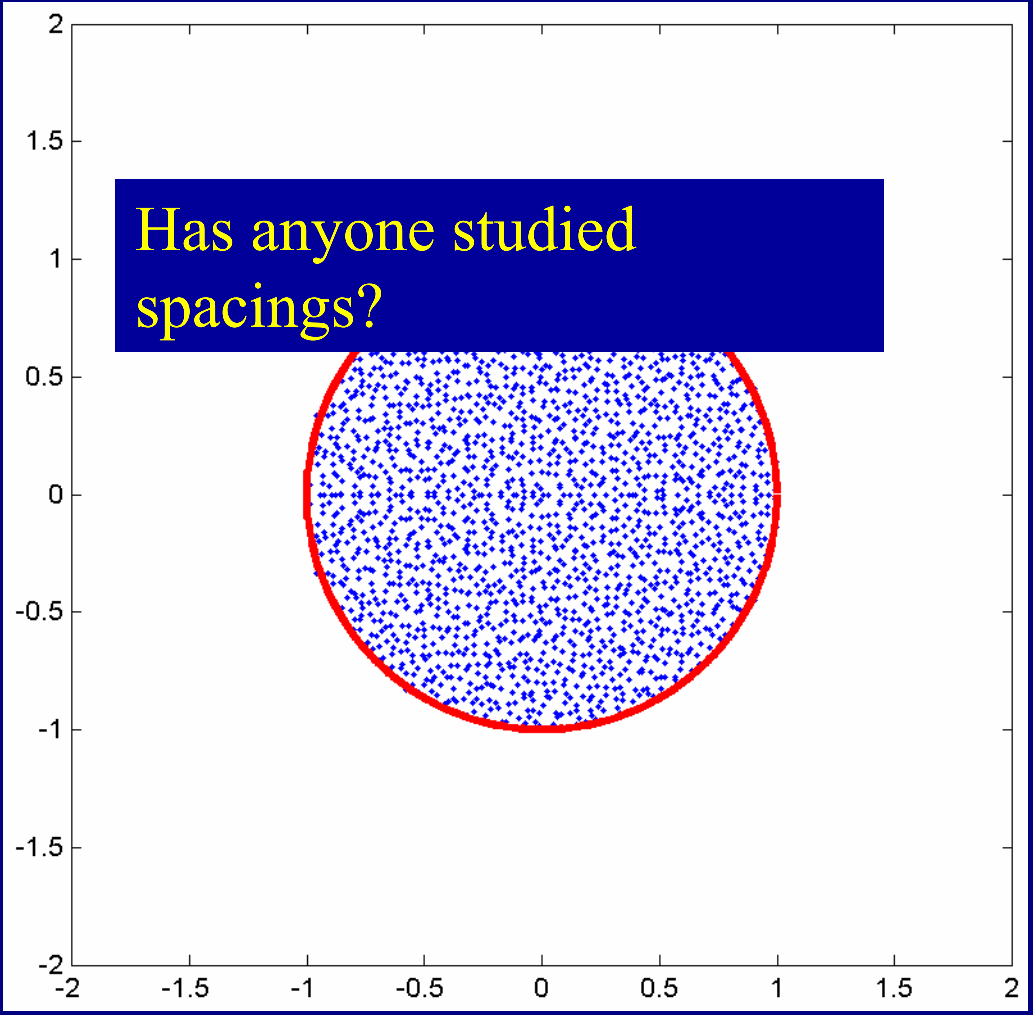
Message

- Ingredient: Take Any important mathematics
- Then Randomize!
- This will have many applications!

Some fun tidbits

- The circular law
- The semi-circular law
- Infinite vs finite
- How many are real?
- Stochastic Numerical Algorithms
- Condition Numbers
- Small networks
- Riemann Zeta Function
- Matrix Jacobians

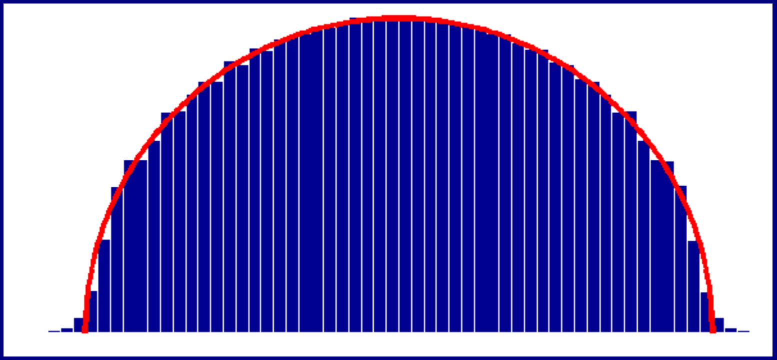
Girko's Circular Law, n=2000



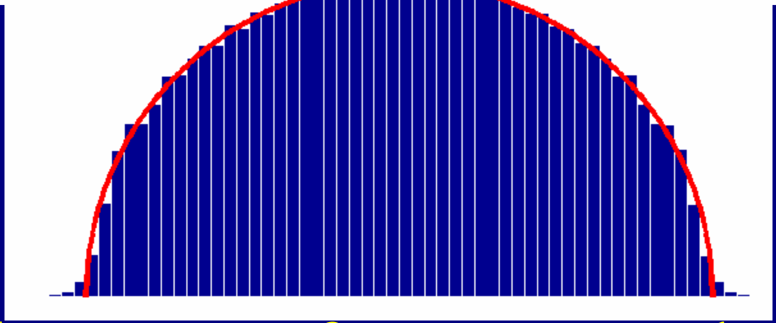
Has anyone studied spacings?

Wigner's Semi-Circle

- The classical & most famous rand eig theorem
- Let S = random symmetric Gaussian
  - MATLAB: A=randn(n); S=(A+A')/2;
- Normalized eigenvalue histogram is a semi-circle
  - Precise statements require n→∞ etc.



Wigner’s Semi-Circle



- The classical & most famous rand eig theorem
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- Normalized eigenvalue histogram is a semi-circle
  - Precise statements require n→∞ etc.

```
n=20;  s=30000; d=.05;     %matrix size, samples, sample dist
e=[];       %gather up eigenvalues
im=1;     %imaginary(1) or real(0)
for i=1:s,
  a=randn(n)+im*sqrt(-1)*randn(n);a=(a+a')/(2*sqrt(2*n*(im+1)));
  v=eig(a)'; e=[e v];
end
hold off; [m x]=hist(e,-1.5:d:1.5); bar(x,m*pi/(2*d*n*s));
axis('square'); axis([-1.5 1.5 -1 2]); hold on;
t=-1:.01:1; plot(t,sqrt(1-t.^2),'r');
```

Elements of Wigner's Proof

- Compute $E(A^{2k})_{11} = \text{mean}(\lambda^{2k}) = (2k)\text{th moment}$
- Verify that the semicircle is the only distribution with these moments
- $(A^{2k})_{11} = \Sigma A_{1x}A_{xy}\ldots A_{wz}A_{z1}$ "paths" of length 2k
- Need only count number of special paths of length 2k on k objects (all other terms 0 or negligible!)
- This is a Catalan Number!

Catalan Numbers

$C_n = \frac{1}{n+1}\binom{2n}{n}$ = # ways to “parenthesize” (n+1) objects

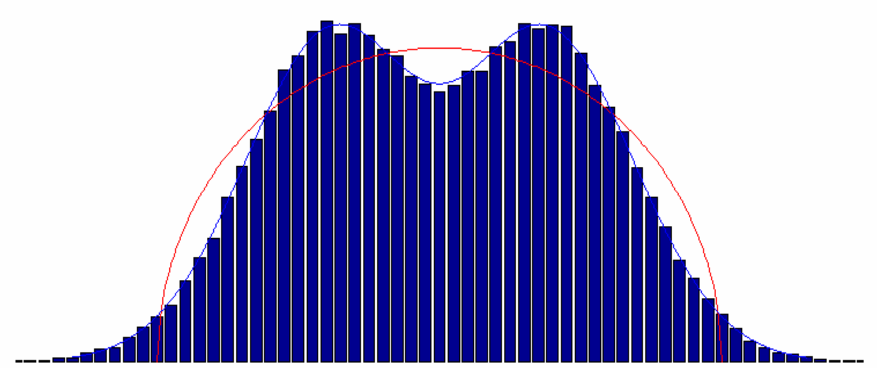
	Matrix Power Term	Graph
(1((23)4))	$A_{12}A_{23}A_{32}A_{24}A_{42}A_{21}$	• • • •
(((12)3)4)	$A_{12}A_{21}A_{13}A_{31}A_{14}A_{41}$	• • • •
(1(2(34)))	$A_{12}A_{23}A_{34}A_{43}A_{32}A_{21}$	• • • •
((12)(34))	$A_{12}A_{21}A_{13}A_{34}A_{43}A_{31}$	• • • •
((1(23))4)	$A_{12}A_{23}A_{32}A_{21}A_{14}A_{41}$	• • • •

= number of special paths on n departing from 1 once

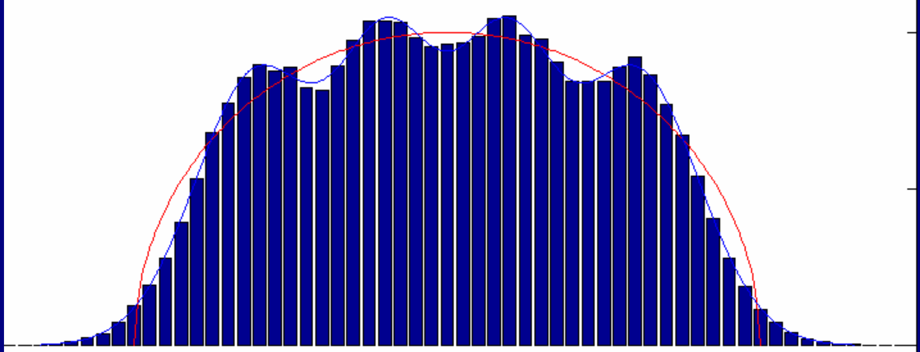
- Pass 1, (load=advance, multiply=retreat), Return to 1

Finite Versions

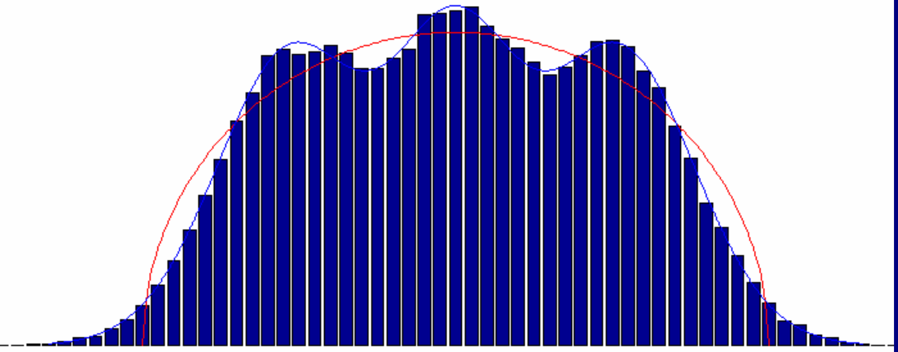
n=2;



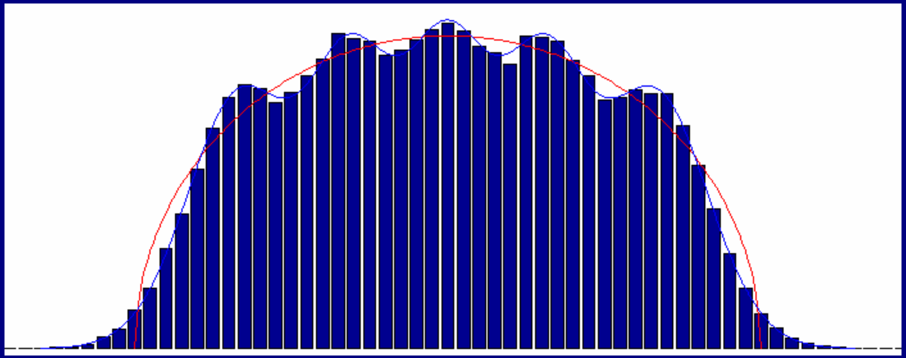
n=4;



n=3;



n=5;



How many eigenvalues of a random matrix are real?

```
>> e=eig(randn(7))
e =
   1.9771
   1.3442
   0.6316
 -1.1664 + 1.3504i
 -1.1664 - 1.3504i
 -2.1461 + 0.7288i
 -2.1461 - 0.7288i
```

3 real

```
>> e=eig(randn(7))
e =
 -2.0767 + 1.1992i
 -2.0767 - 1.1992i
  2.9437
  0.0234 + 0.4845i
  0.0234 - 0.4845i
  1.1914 + 0.3629i
  1.1914 - 0.3629i
```

1 real

```
>> e=eig(randn(7))
e =
 -2.1633
 -0.9264
 -0. 3283
  2.5242
  1.6230 + 0.9011i
  1.6230 - 0.9011i
  0.5467
```

5 real

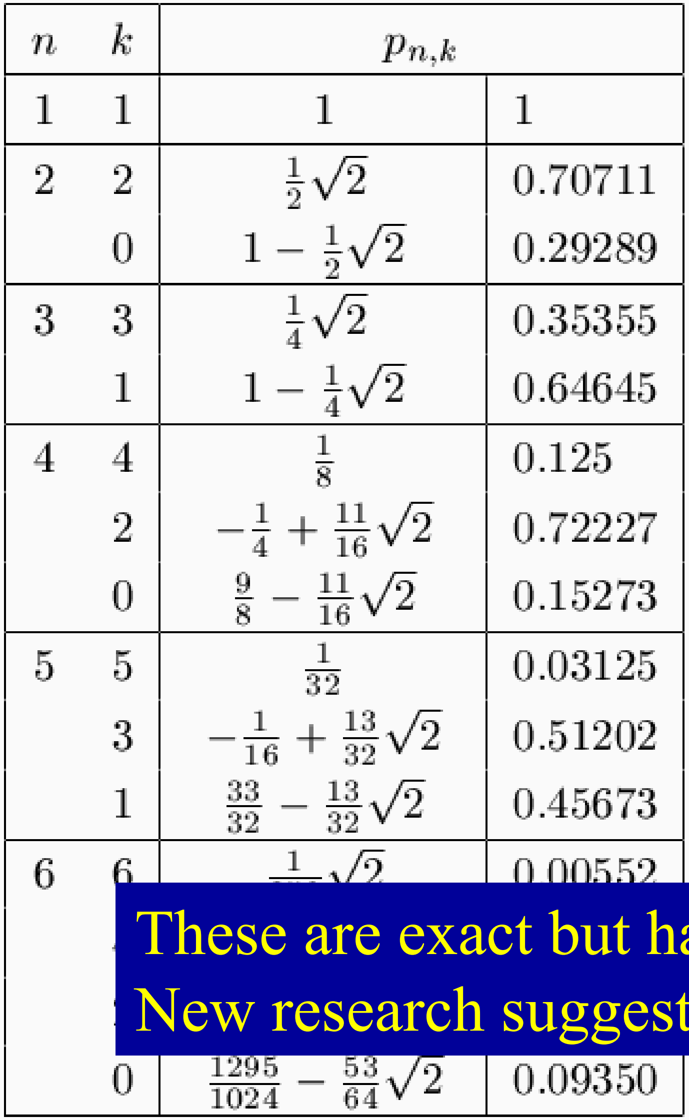
7x7 random Gaussian

How many eigenvalues of a random matrix are real?

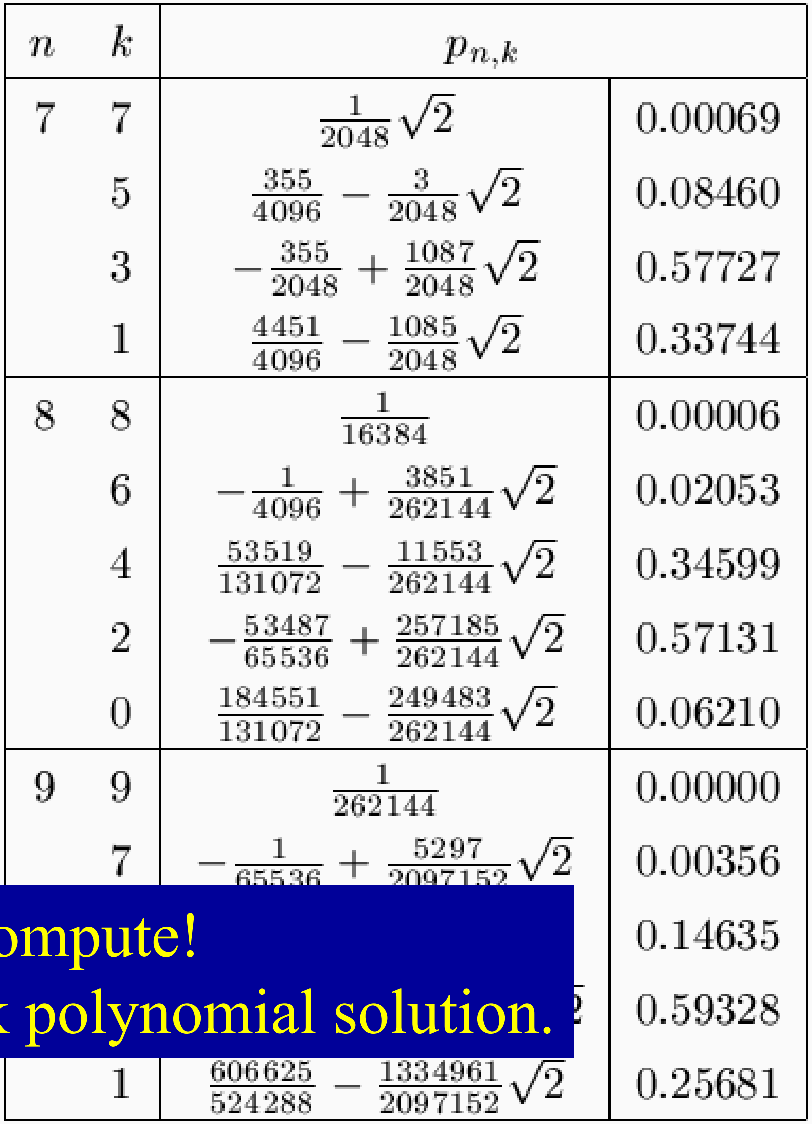
n=7

7 reals	$\frac{1}{2048}\sqrt{2}$	0.00069
5 reals	$\frac{355}{4096}-\frac{3}{2048}\sqrt{2}$	0.08460
3 reals	$-\frac{355}{2048}+\frac{1087}{2048}\sqrt{2}$	0.57727
1 real	$\frac{4451}{4096}-\frac{1085}{2048}\sqrt{2}$	0.33744

How many eigenvalues of a random



n	k	$p_{n,k}$	
1	1	1	1
2	2	$\frac{1}{2}\sqrt{2}$	0.70711
	0	$1-\frac{1}{2}\sqrt{2}$	0.29289
3	3	$\frac{1}{4}\sqrt{2}$	0.35355
	1	$1-\frac{1}{4}\sqrt{2}$	0.64645
4	4	$\frac{1}{8}$	0.125
	2	$-\frac{1}{4}+\frac{11}{16}\sqrt{2}$	0.72227
	0	$\frac{9}{8}-\frac{11}{16}\sqrt{2}$	0.15273
5	5	$\frac{1}{32}$	0.03125
	3	$-\frac{1}{16}+\frac{13}{32}\sqrt{2}$	0.51202
	1	$\frac{33}{32}-\frac{13}{32}\sqrt{2}$	0.45673
6	6	$\frac{1}{[illegible]}\sqrt{2}$	0.00552
	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]	[illegible]
	0	$\frac{1295}{1024}-\frac{53}{64}\sqrt{2}$	0.09350



n	k	$p_{n,k}$	
7	7	$\frac{1}{2048}\sqrt{2}$	0.00069
	5	$\frac{355}{4096}-\frac{3}{2048}\sqrt{2}$	0.08460
	3	$-\frac{355}{2048}+\frac{1087}{2048}\sqrt{2}$	0.57727
	1	$\frac{4451}{4096}-\frac{1085}{2048}\sqrt{2}$	0.33744
8	8	$\frac{1}{16384}$	0.00006
	6	$-\frac{1}{4096}+\frac{3851}{262144}\sqrt{2}$	0.02053
	4	$\frac{53519}{131072}-\frac{11553}{262144}\sqrt{2}$	0.34599
	2	$-\frac{53487}{65536}+\frac{257185}{262144}\sqrt{2}$	0.57131
	0	$\frac{184551}{131072}-\frac{249483}{262144}\sqrt{2}$	0.06210
9	9	$\frac{1}{262144}$	0.00000
	7	$-\frac{1}{65536}+\frac{5297}{2097152}\sqrt{2}$	0.00356
	[illegible]	[illegible]	0.14635
	[illegible]	[illegible]	0.59328
	1	$\frac{606625}{524288}-\frac{1334961}{2097152}\sqrt{2}$	0.25681

These are exact but hard to compute!

New research suggests a Jack polynomial solution.

How many eigenvalues of a random matrix are real?

- The Probability that a matrix has all real eigenvalues is exactly

$$P_{n,n} = 2^{-n(n-1)/4}$$

Proof based on Schur Form

Gram Schmidt (or QR) Stochastically

- Gram Schmidt
  = Orthogonal Transformations to
  Upper Triangular Form

- A = Q * R (orthog * upper triangular)

Orthogonal Invariance of Gaussians

Q*randn(n,1)

≡

randn(n,1)

If Q orthogonal

$$Q * \begin{bmatrix} G \\ G \\ G \\ G \\ G \\ G \\ G \end{bmatrix}$$

Orthogonal Invariance

Q*randn(n,1)

≡

randn(n,1)

If Q orthogonal

$$Q*\begin{bmatrix}G\\G\\G\\G\\G\\G\\G\end{bmatrix}=\begin{bmatrix}G\\G\\G\\G\\G\\G\\G\end{bmatrix}$$

Chi Distribution

$$\boxed{\begin{array}{c}\text{norm(randn(n,1))} \\ \equiv \\ \chi_n\end{array}}$$

$$\left\| \begin{matrix} G \\ G \\ G \\ G \\ G \\ G \\ G \end{matrix} \right\| = \chi_n$$

Chi Distribution

norm(randn(n,1))

$$\equiv$$

$$\chi_n$$

$$\left\| \begin{matrix} G \\ G \\ G \\ G \\ G \\ G \\ G \end{matrix} \right\| = \chi_n$$

Chi Distribution

norm(randn(n,1))

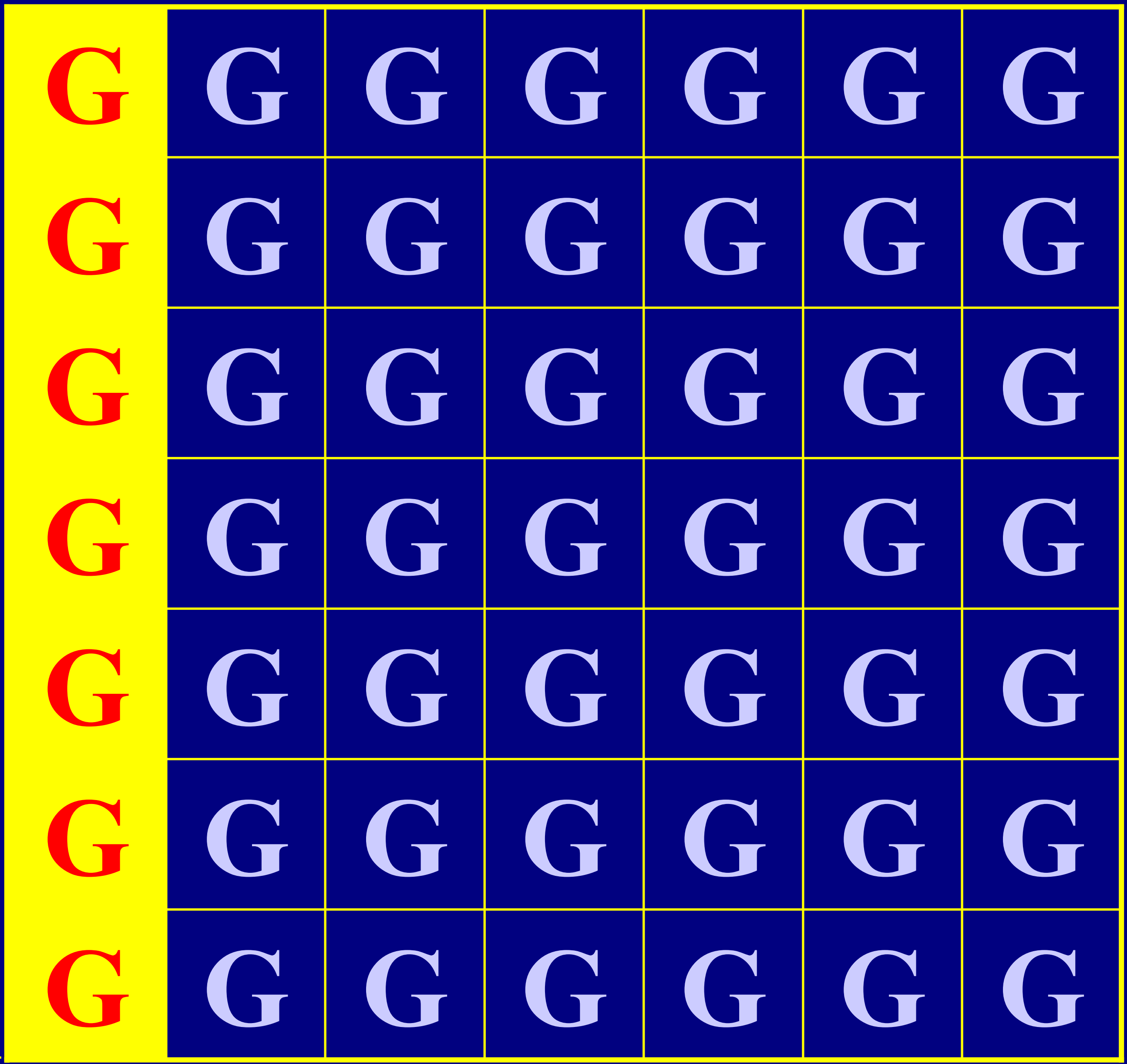
$$\equiv$$

$$\chi_n$$

$$\left\| \begin{matrix} G \\ G \\ G \\ G \\ G \\ G \\ G \end{matrix} \right\| = \chi_n$$

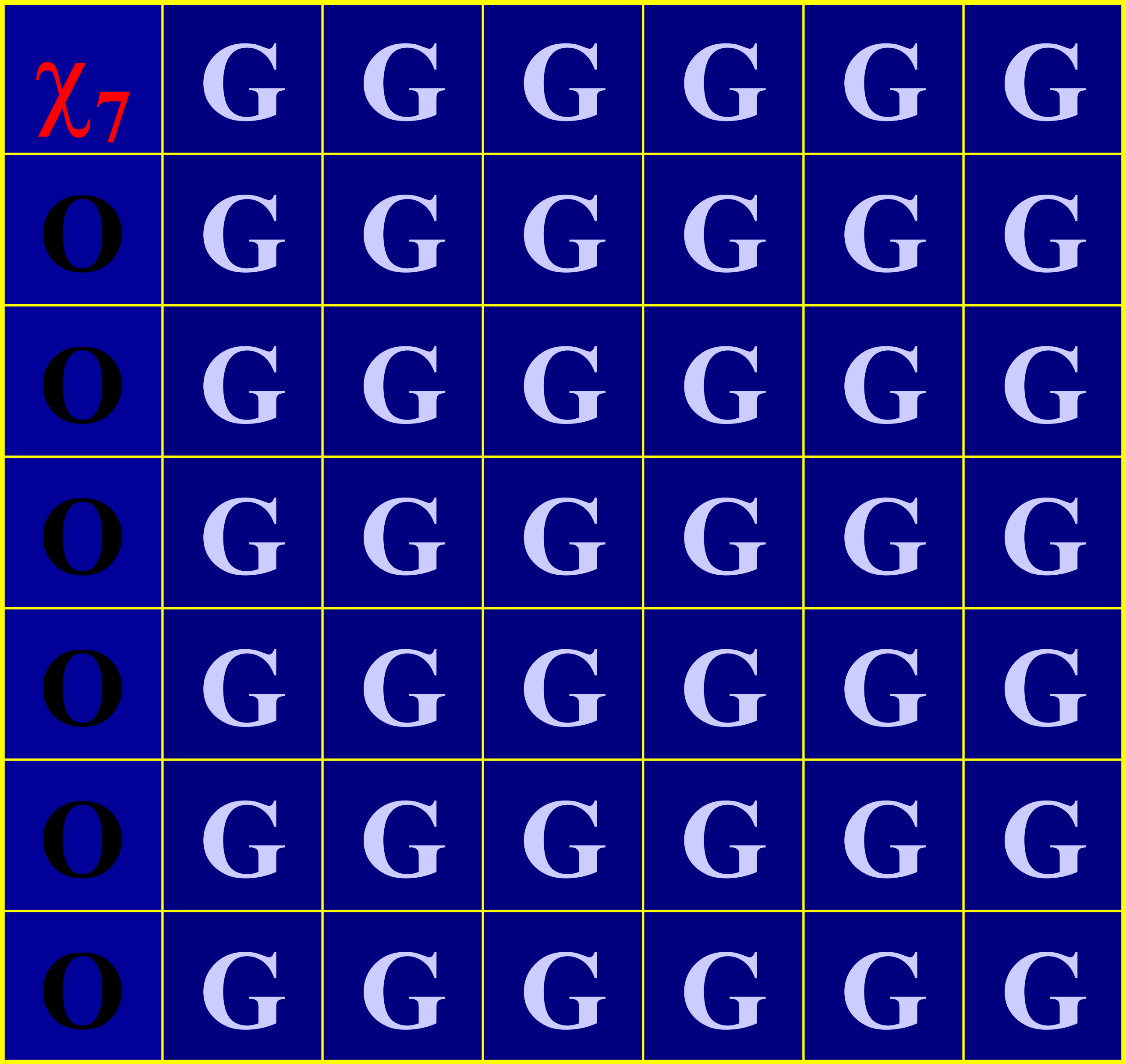
n need not be integer

G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G

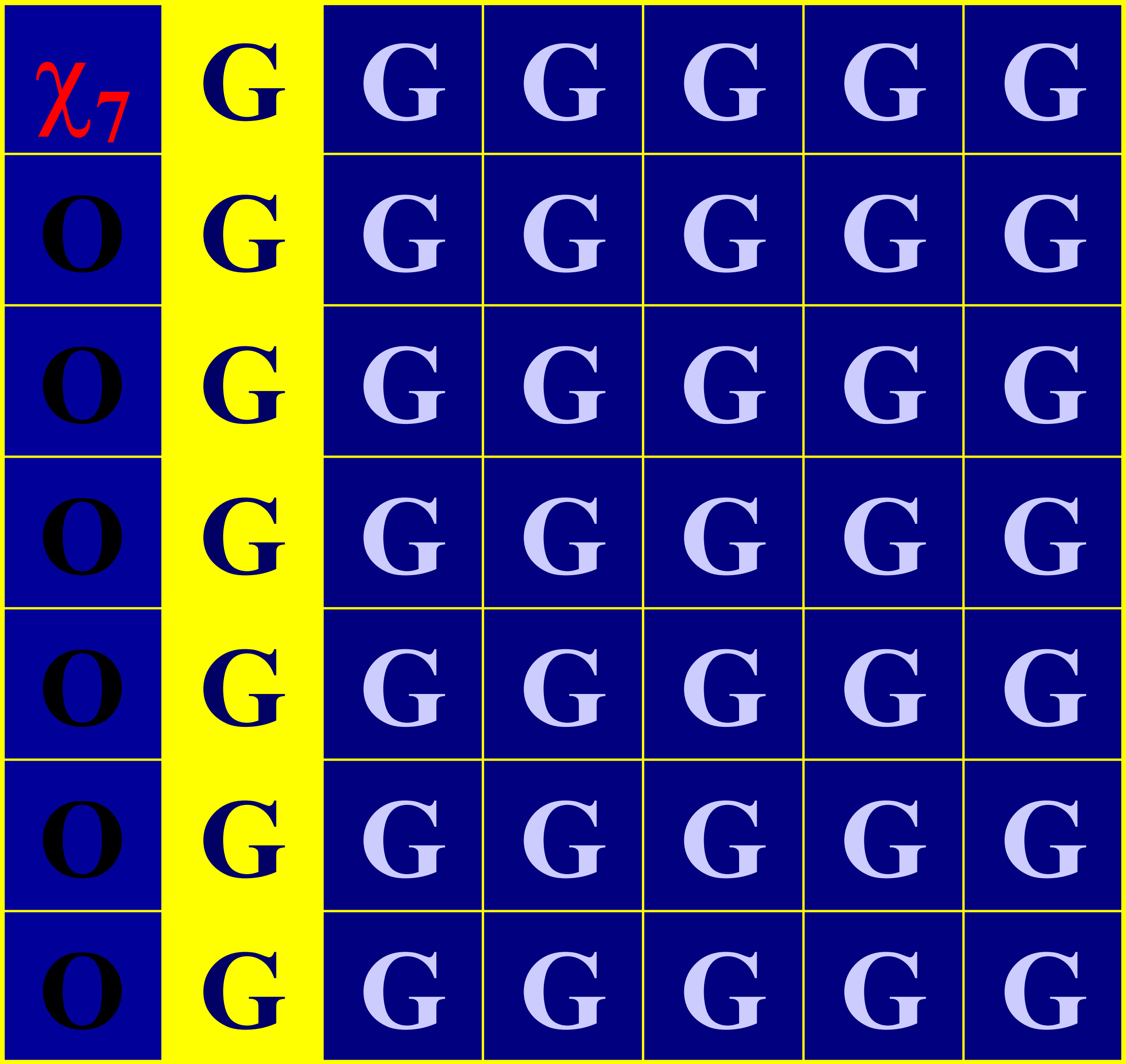


G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G
G	G	G	G	G	G	G

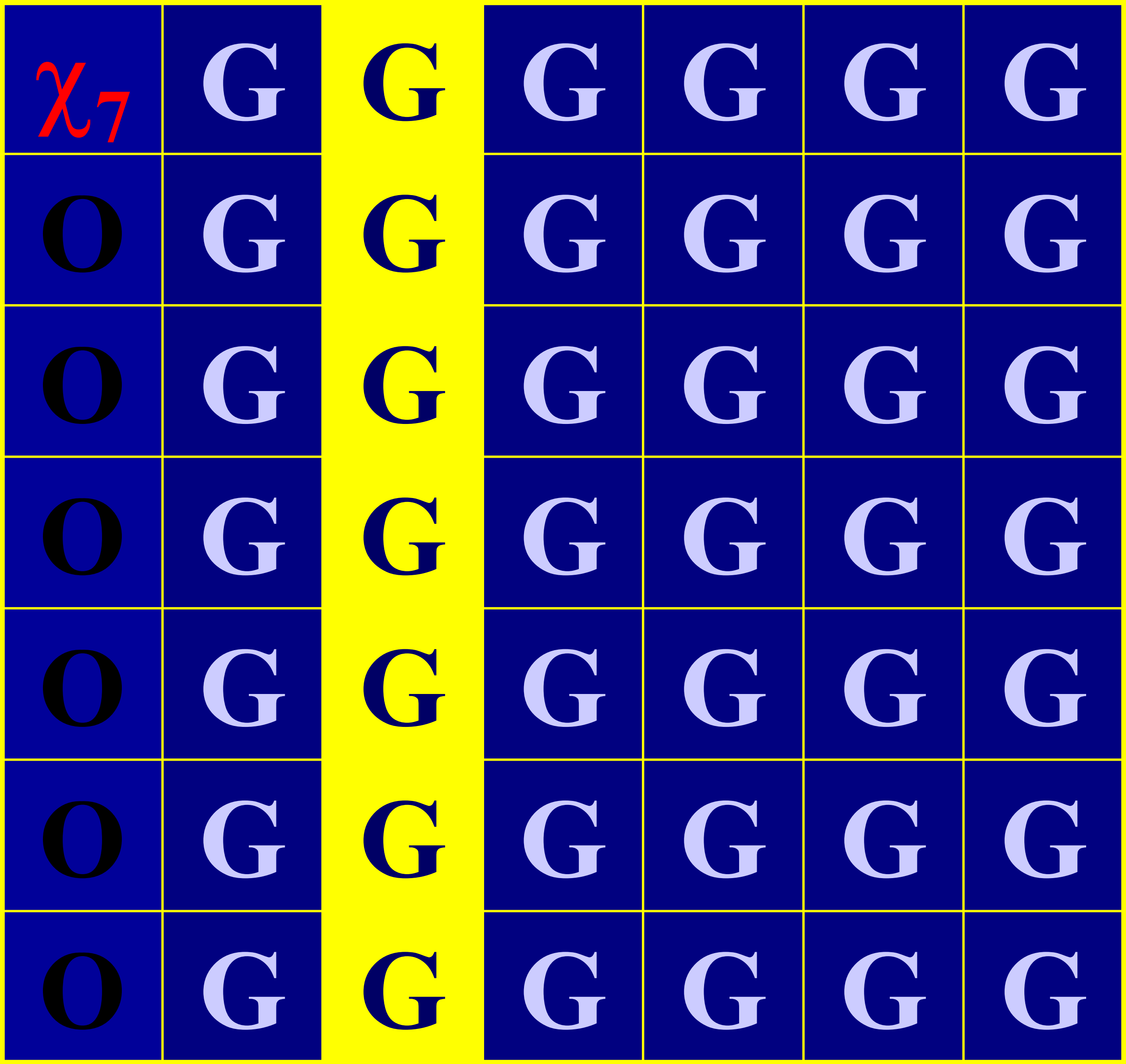
χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G



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O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G

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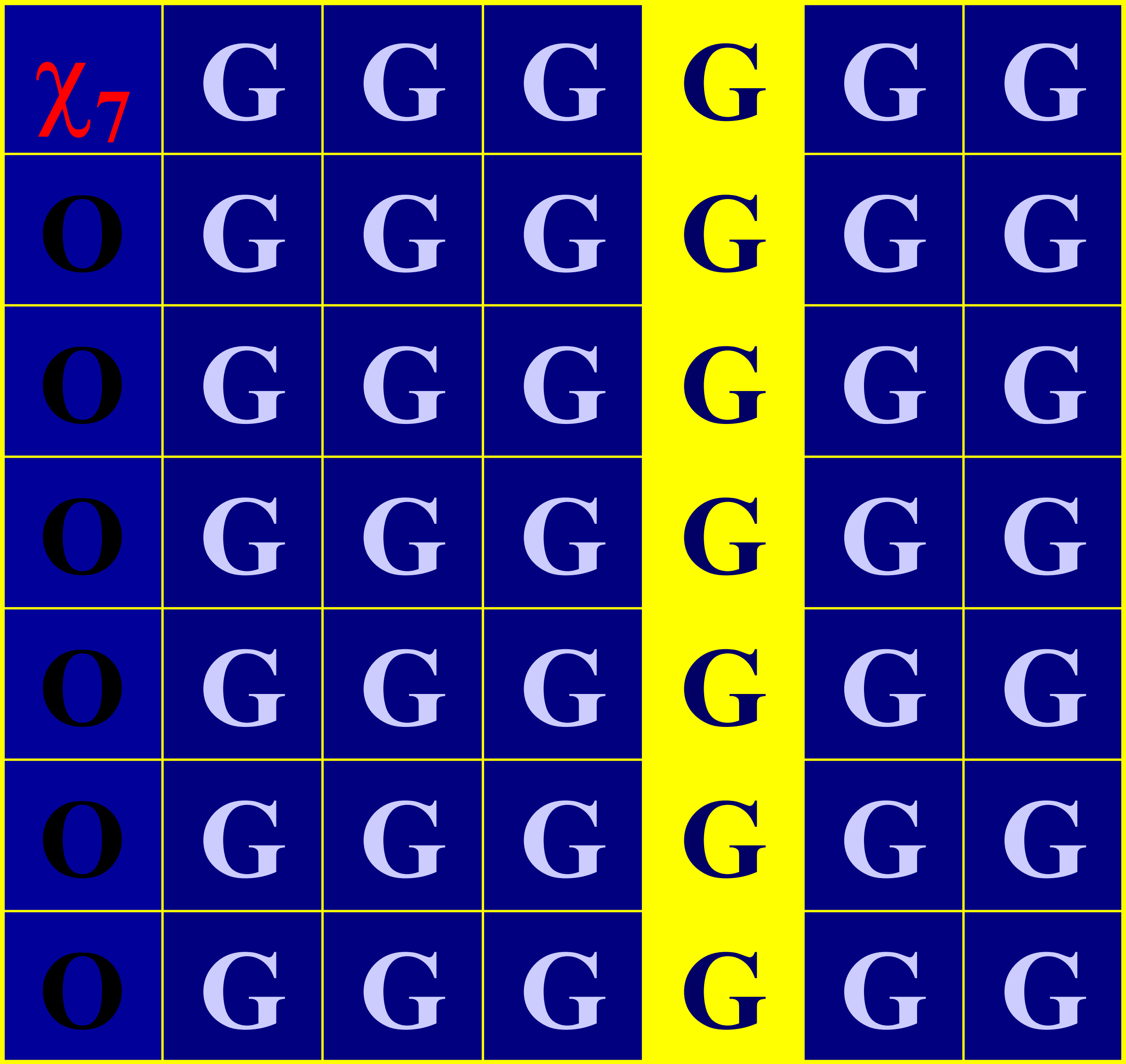
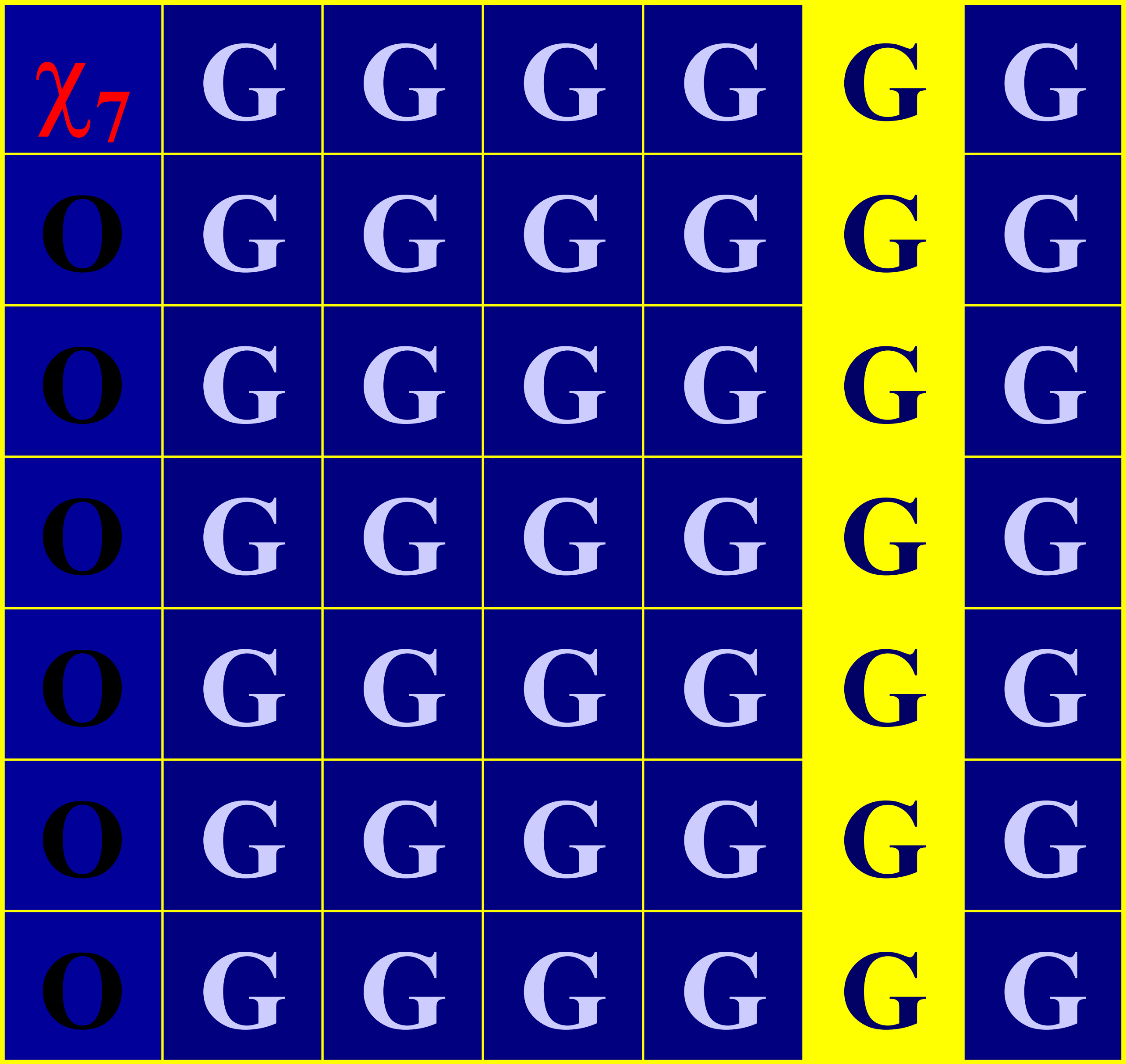
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
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O	G	G	G	G	G	G
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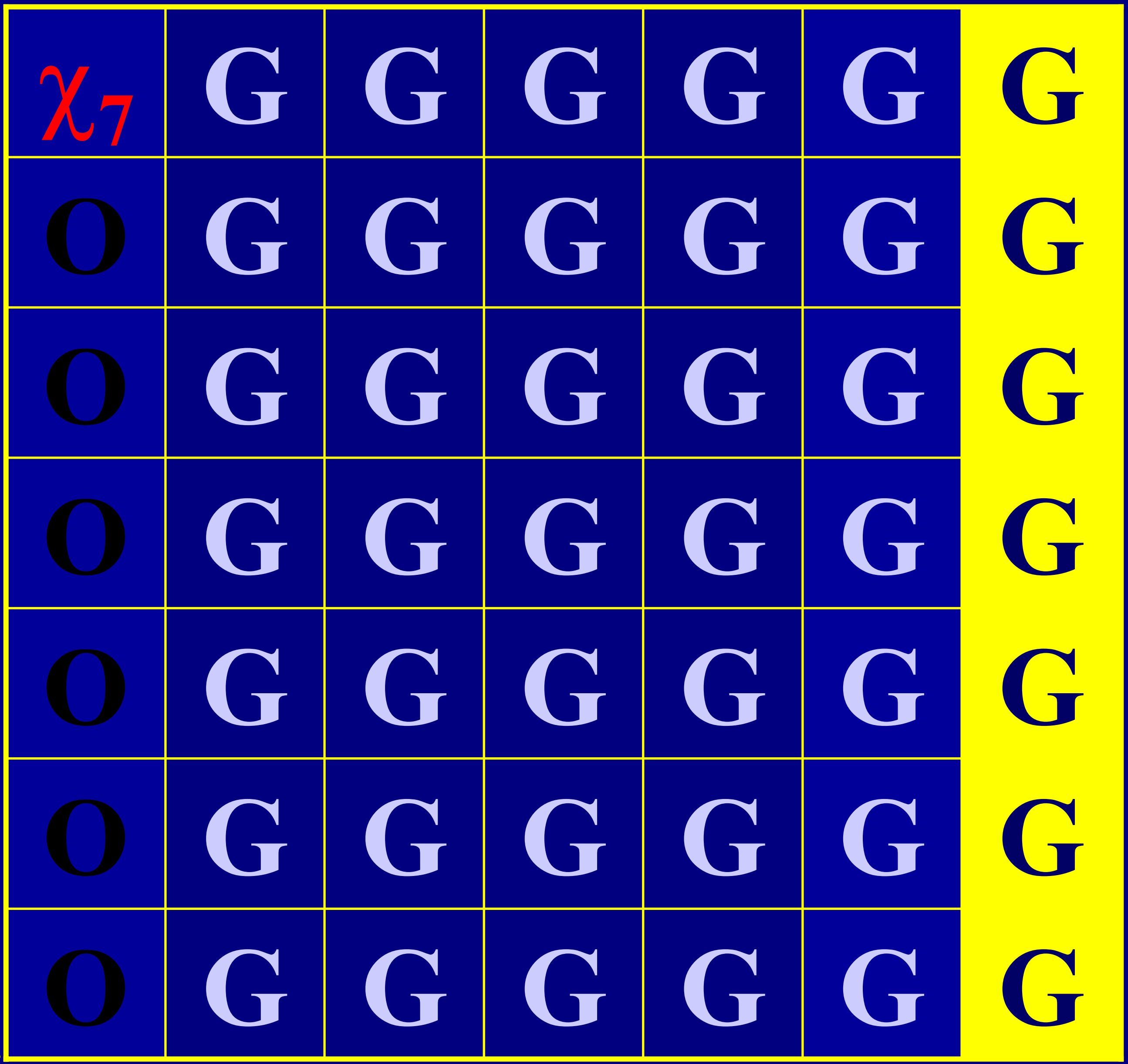
O	G	G	G	G	G	G



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O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G

χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
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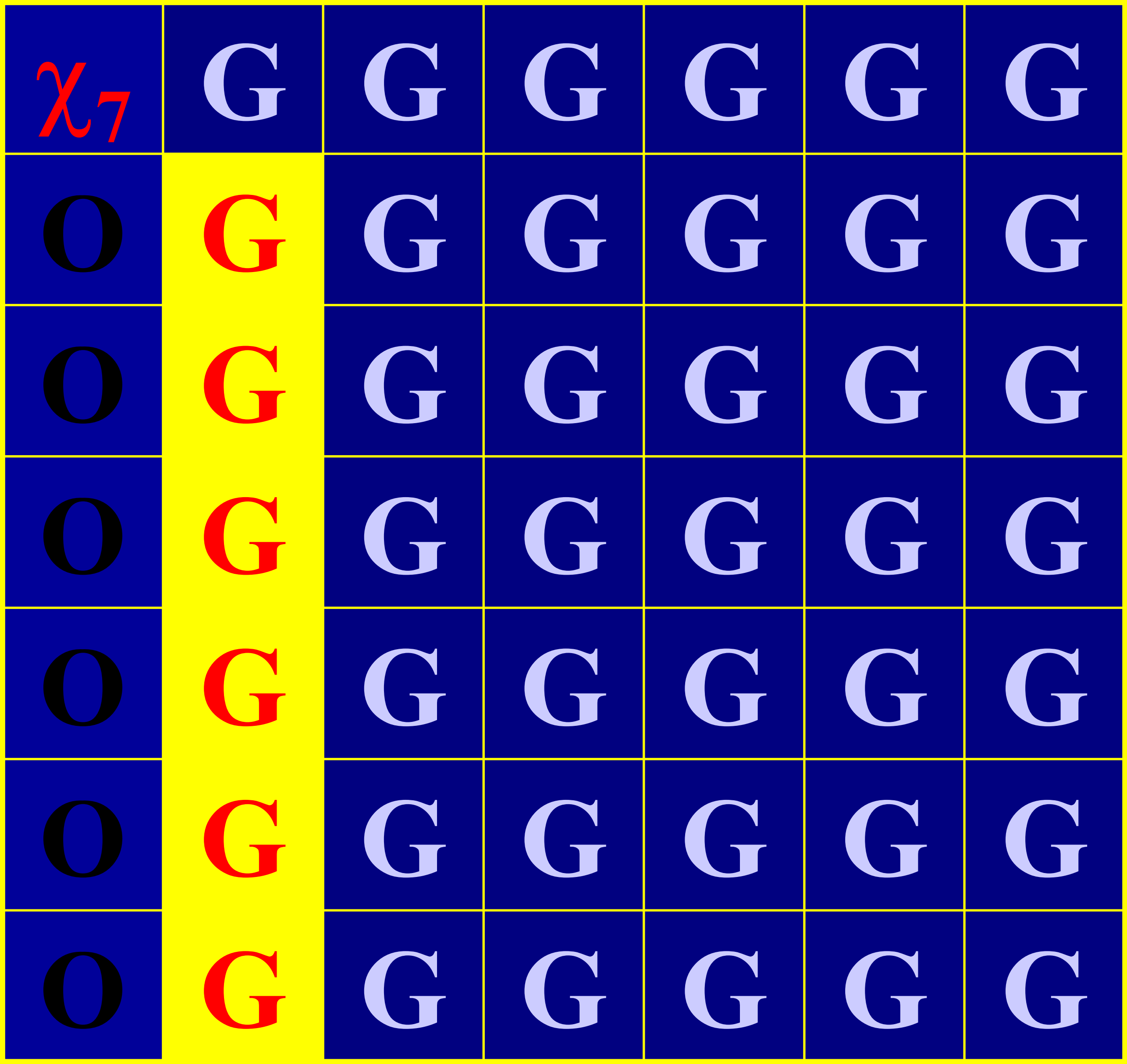
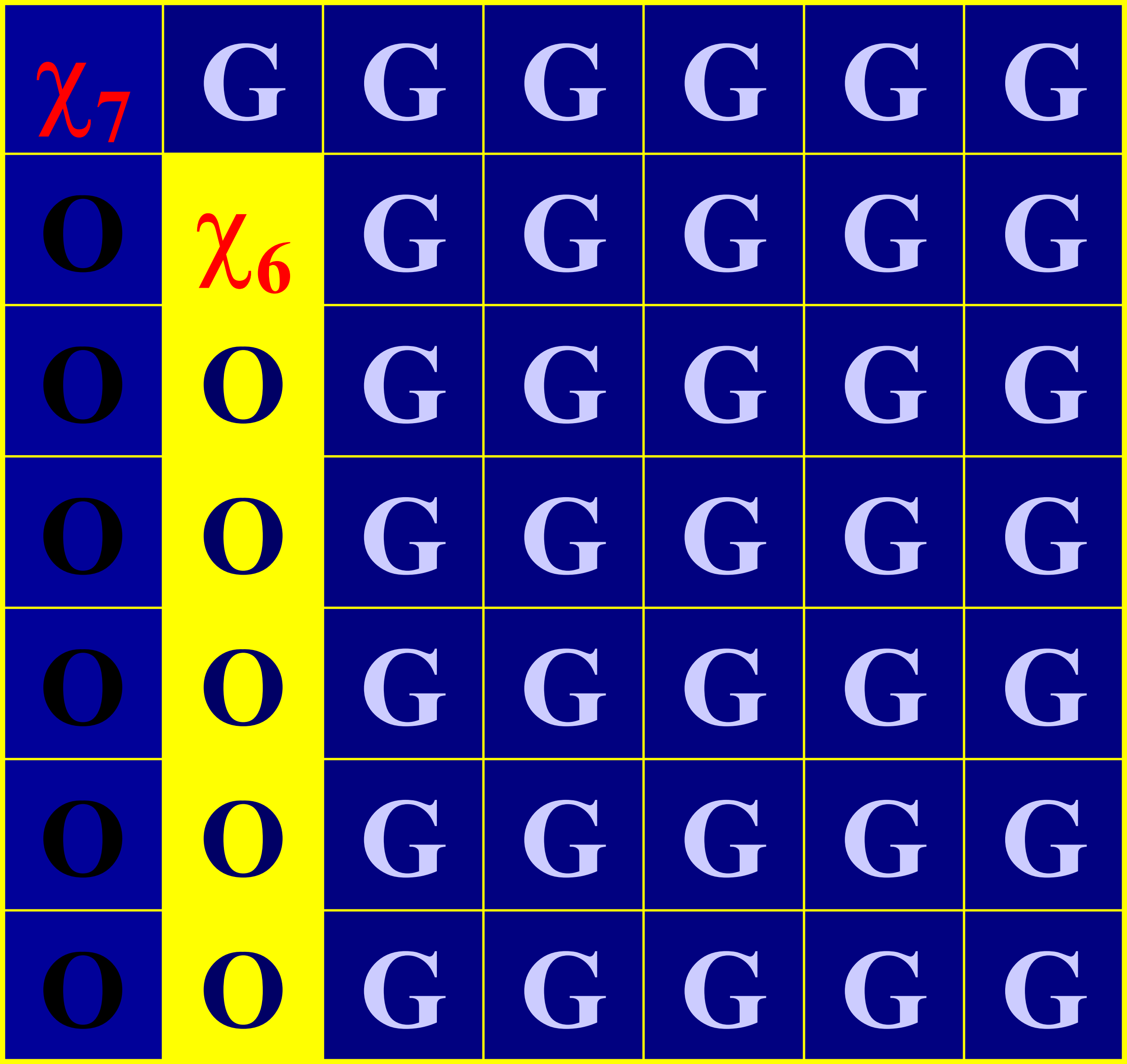
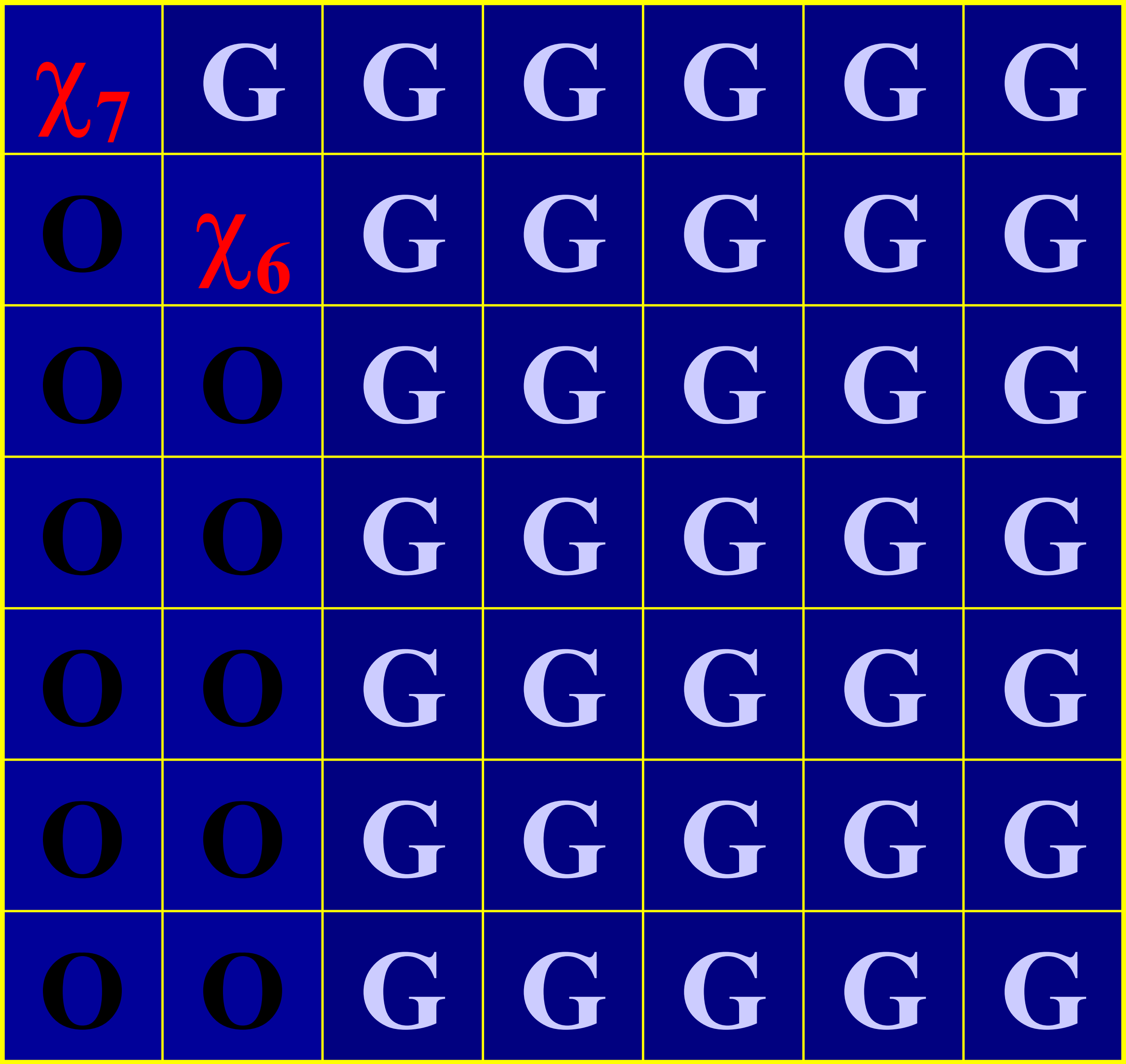
O	G	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G
O	G	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
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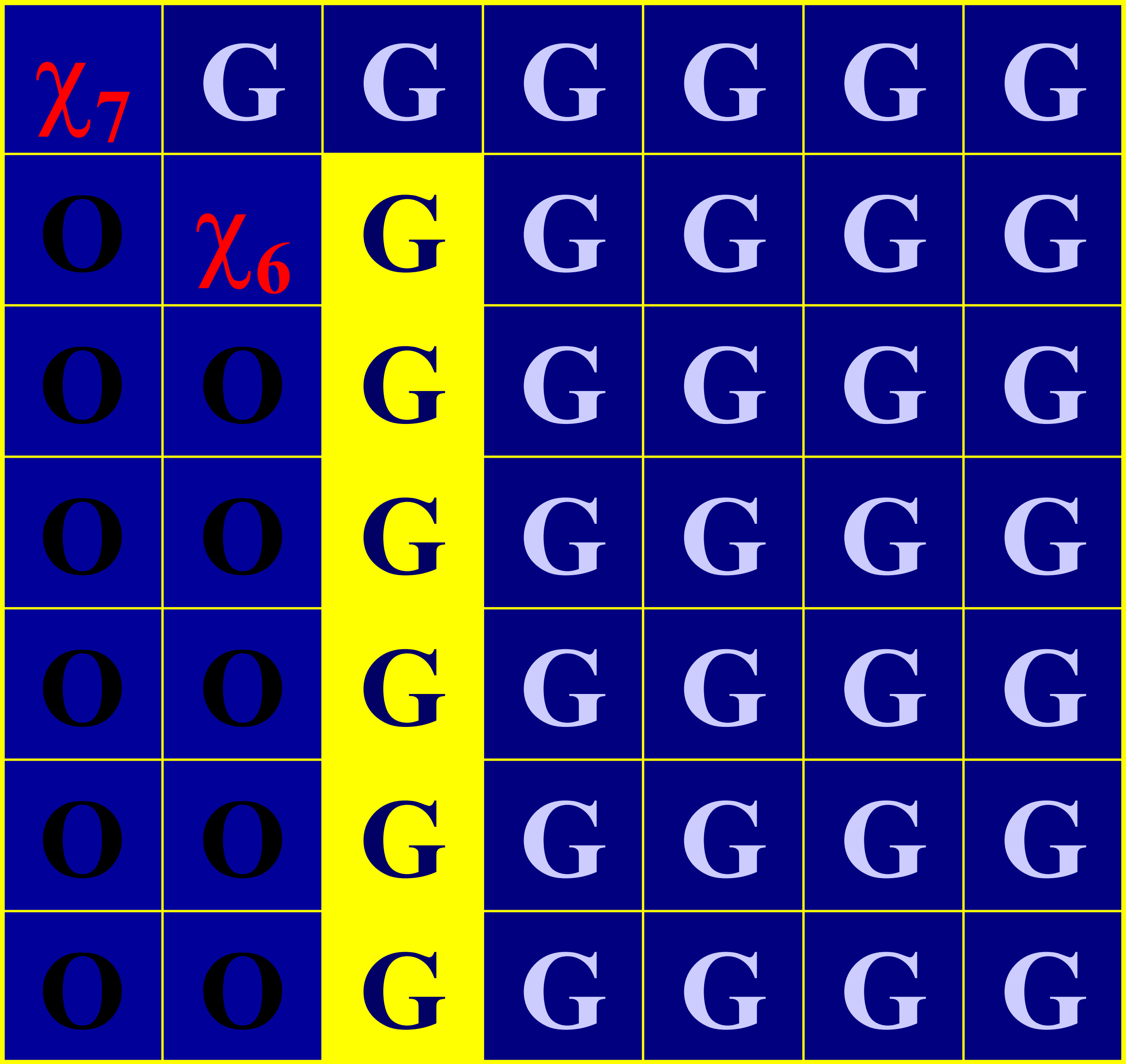
O	O	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	G	G	G	G	G
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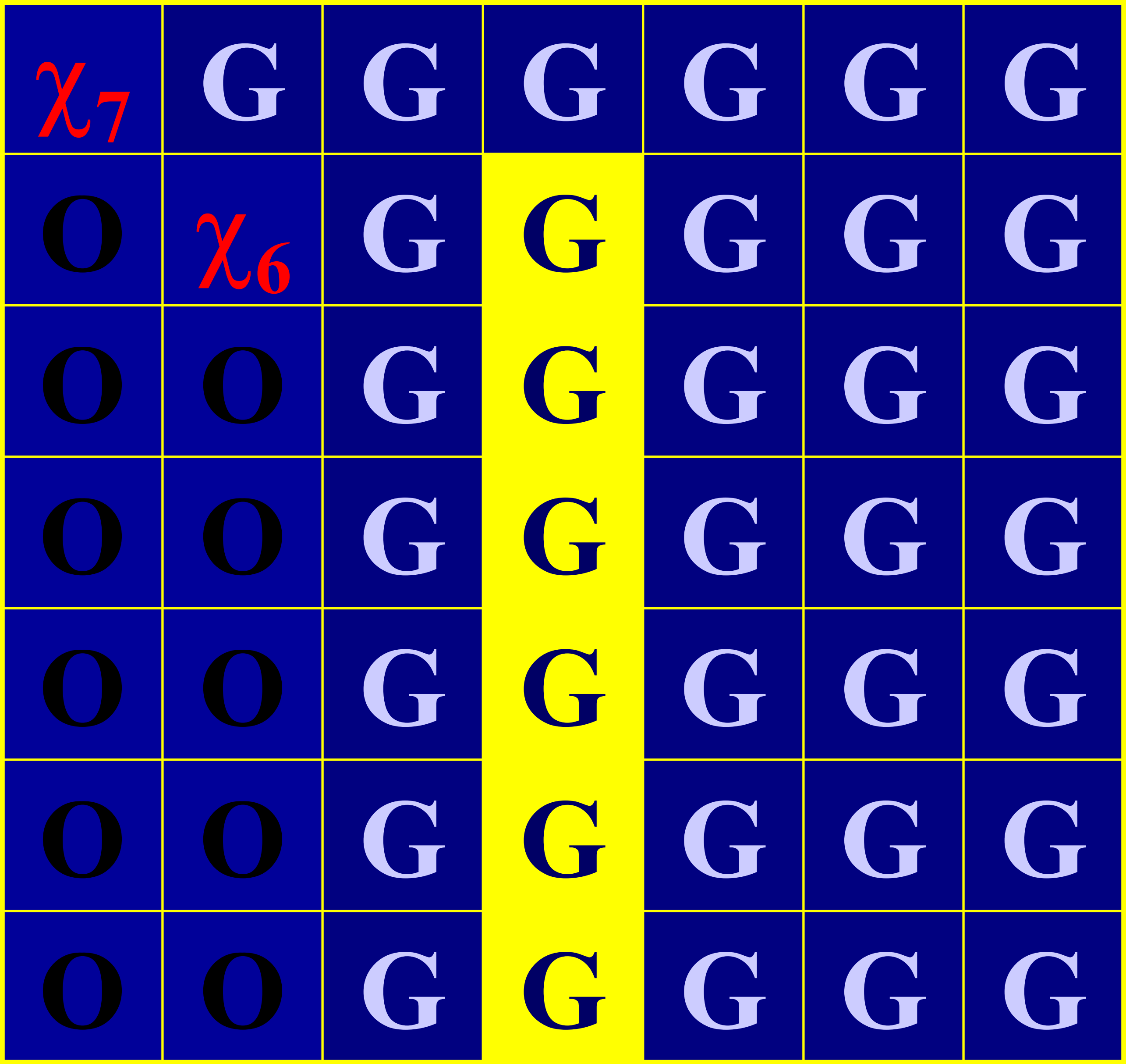
O	O	G	G	G	G	G



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O	χ_6	G	G	G	G	G
O	O	G	G	G	G	G
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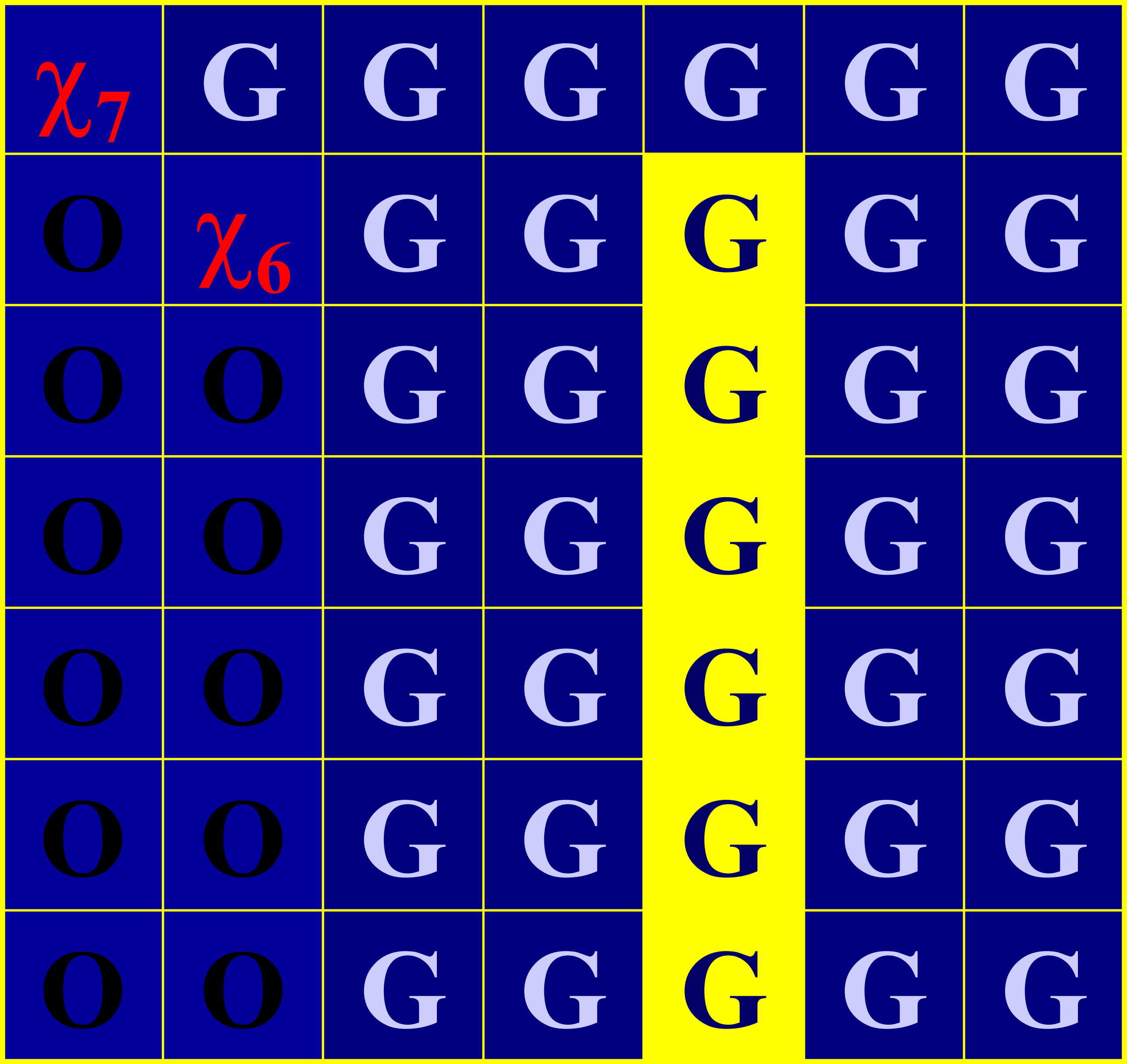
O	O	G	G	G	G	G



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O	O	G	G	G	G	G
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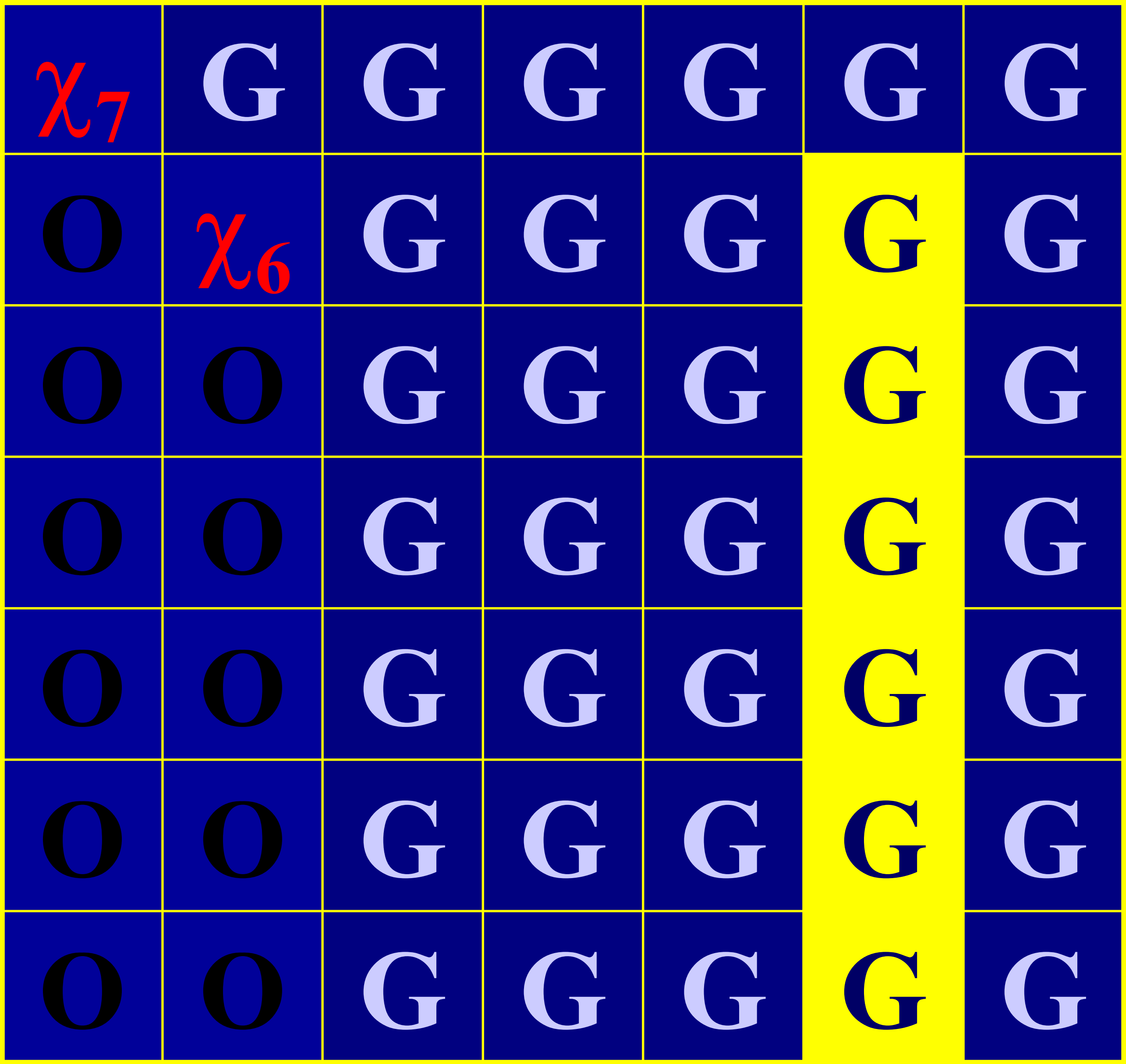
O	O	G	G	G	G	G



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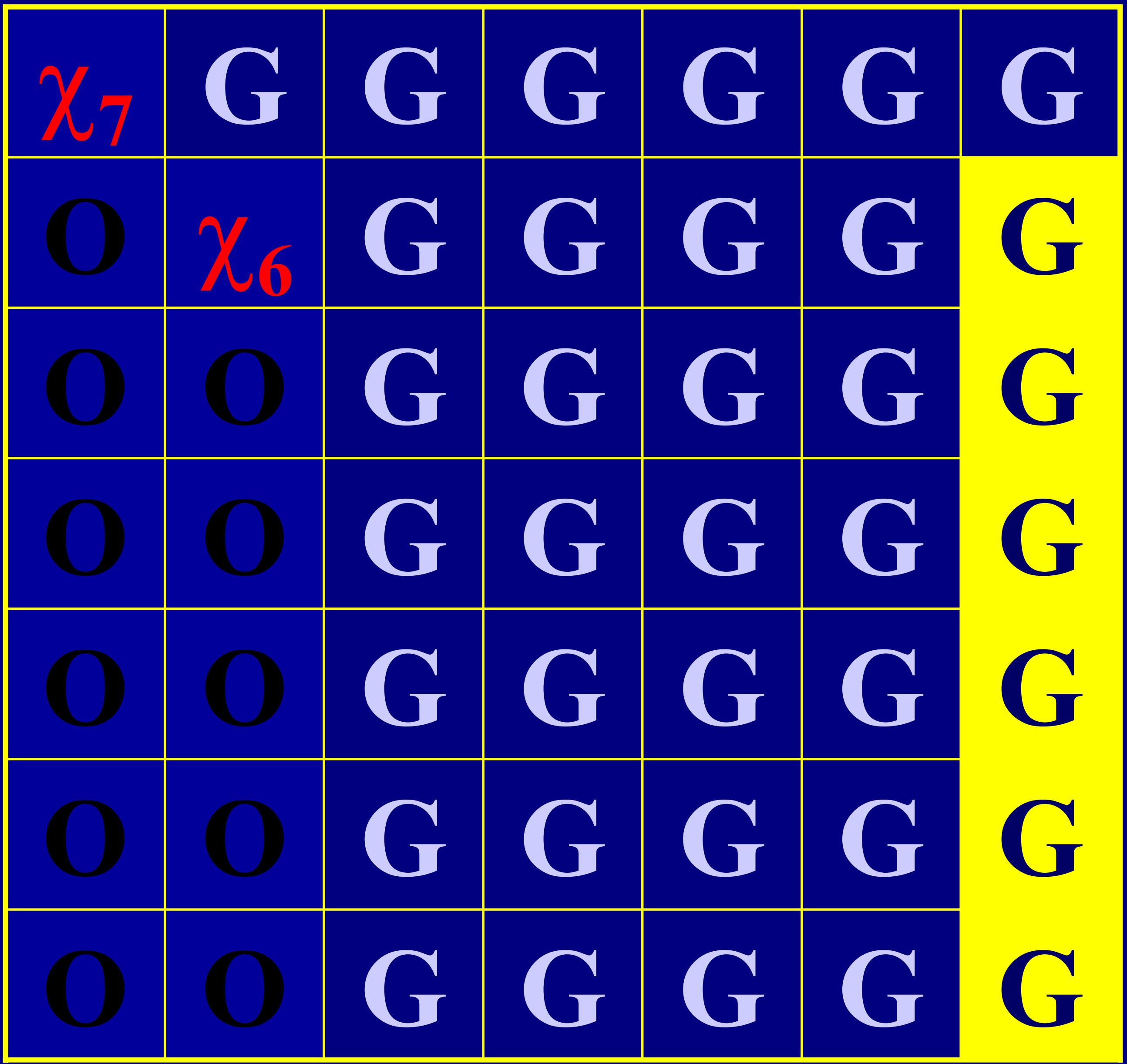
O	χ_6	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G



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O	χ_6	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
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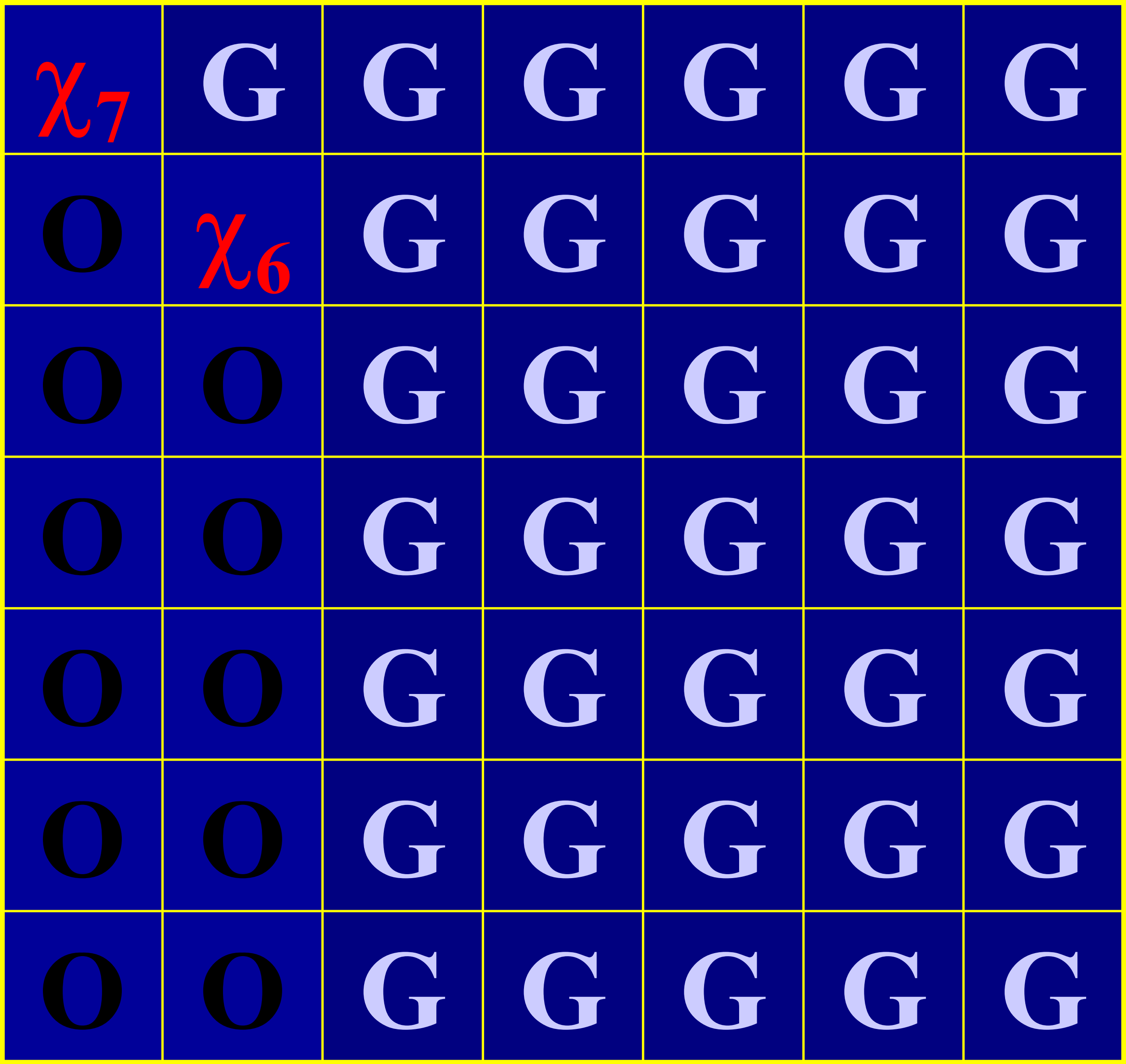
O	O	G	G	G	G	G



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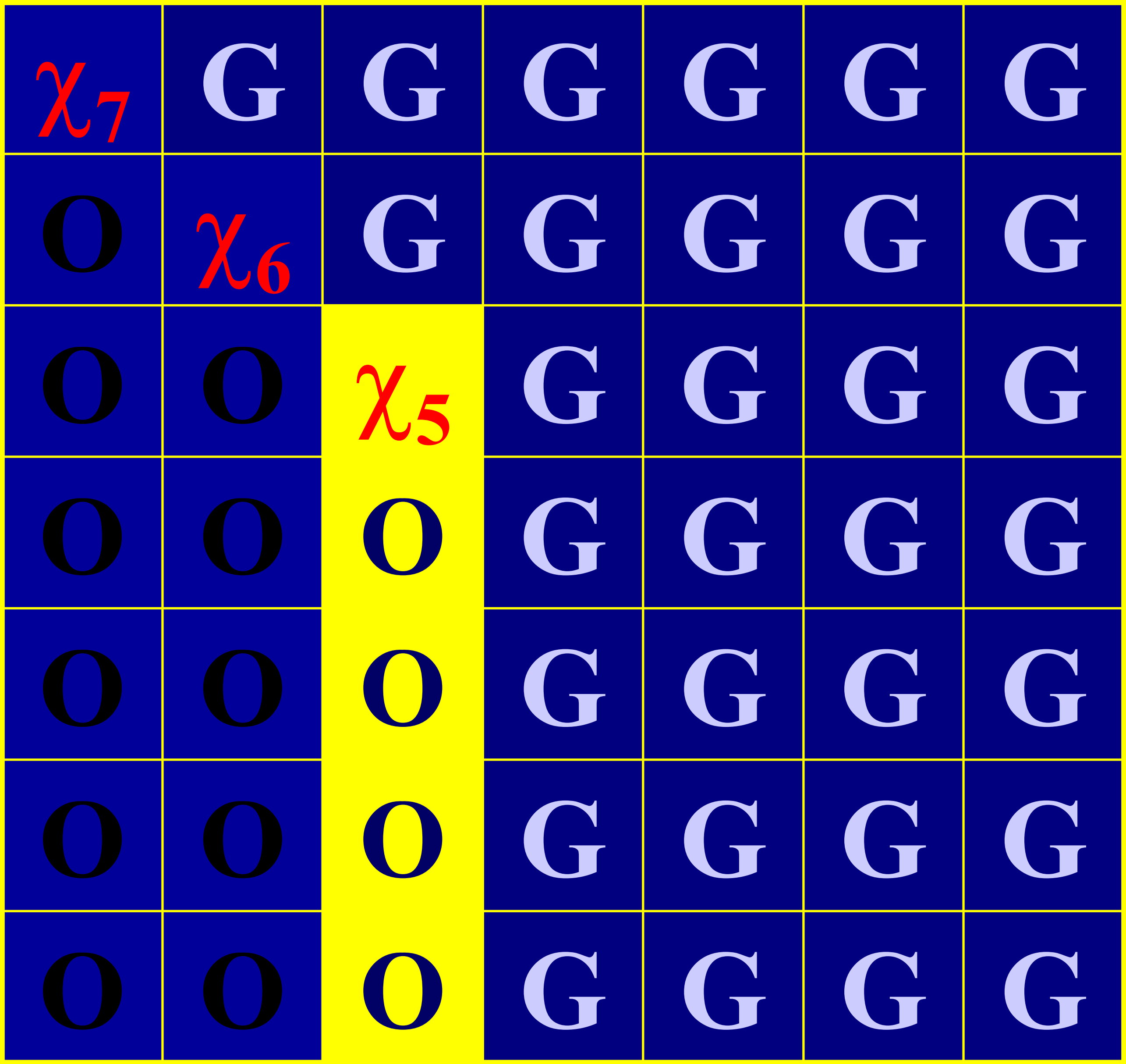
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G
O	O	G	G	G	G	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
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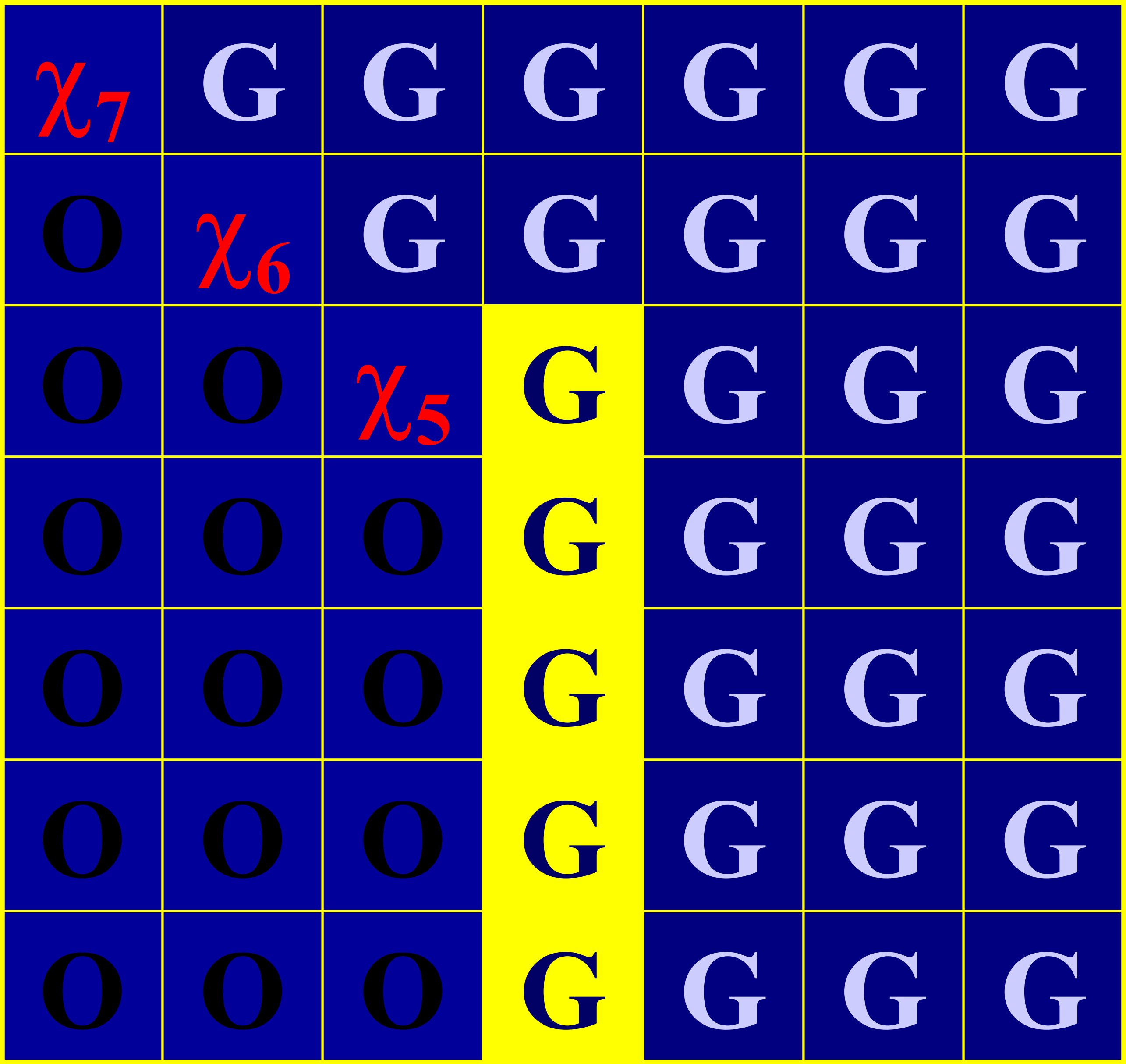
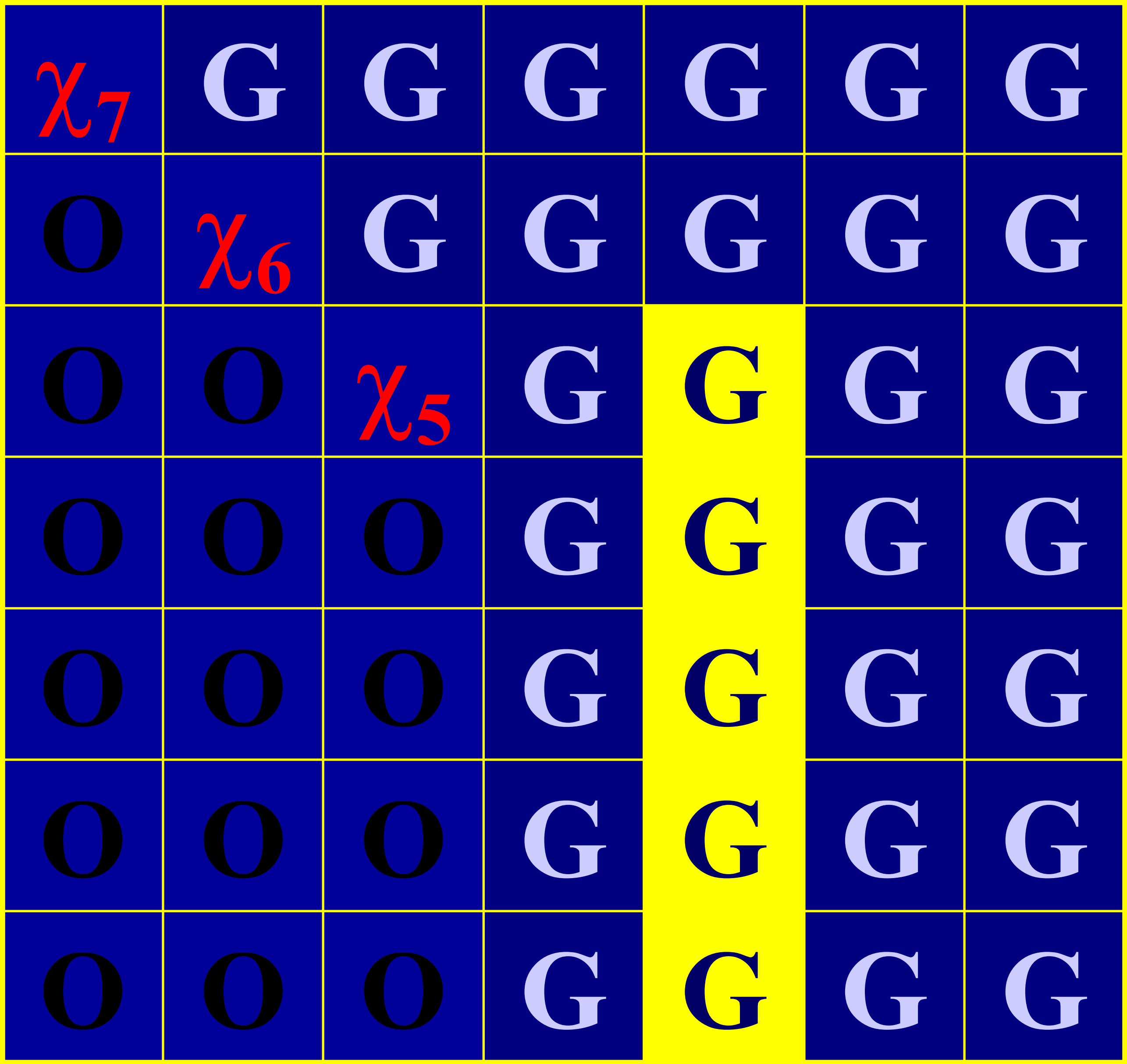
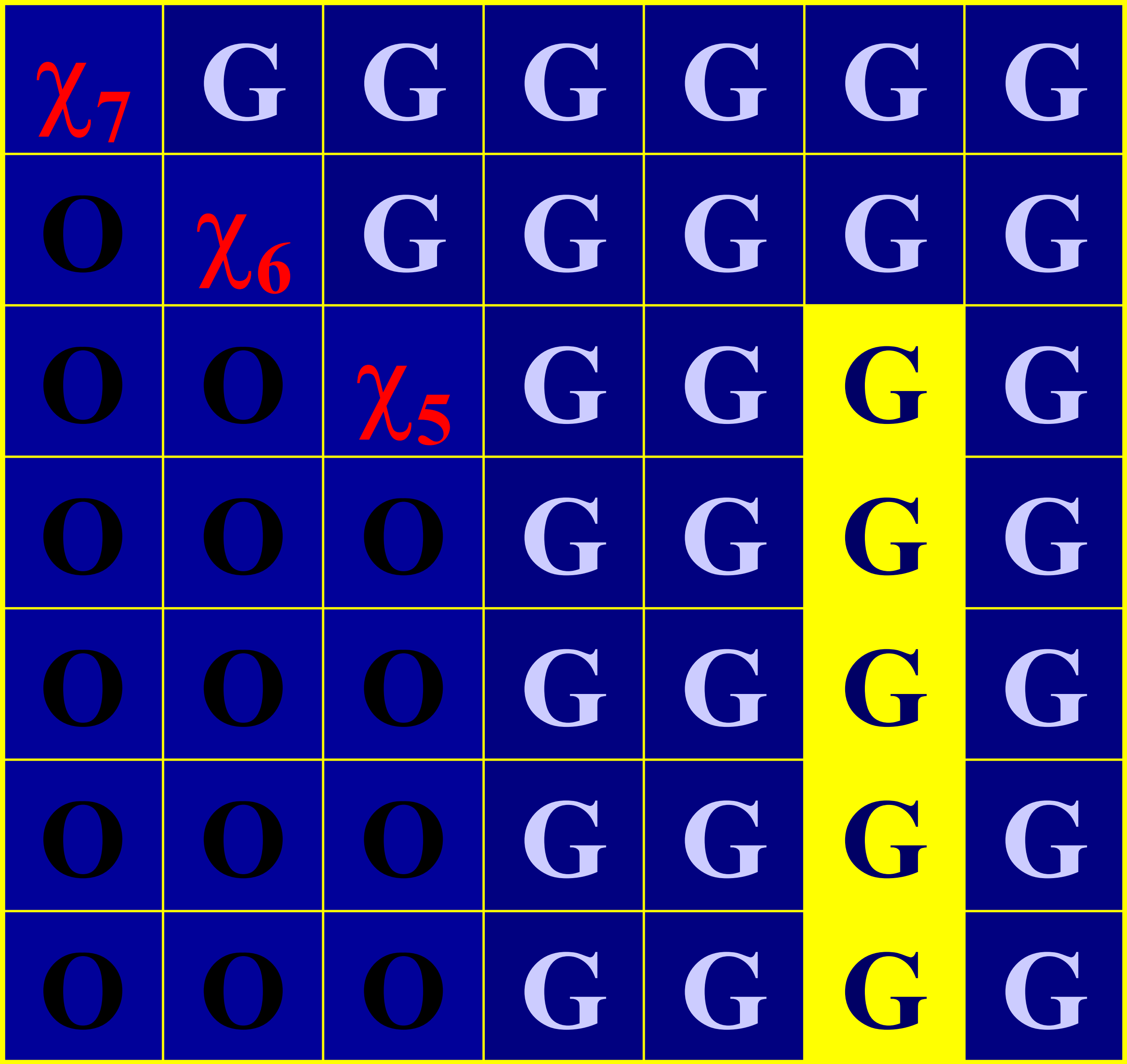
O	O	O	G	G	G	G
O	O	O	G	G	G	G



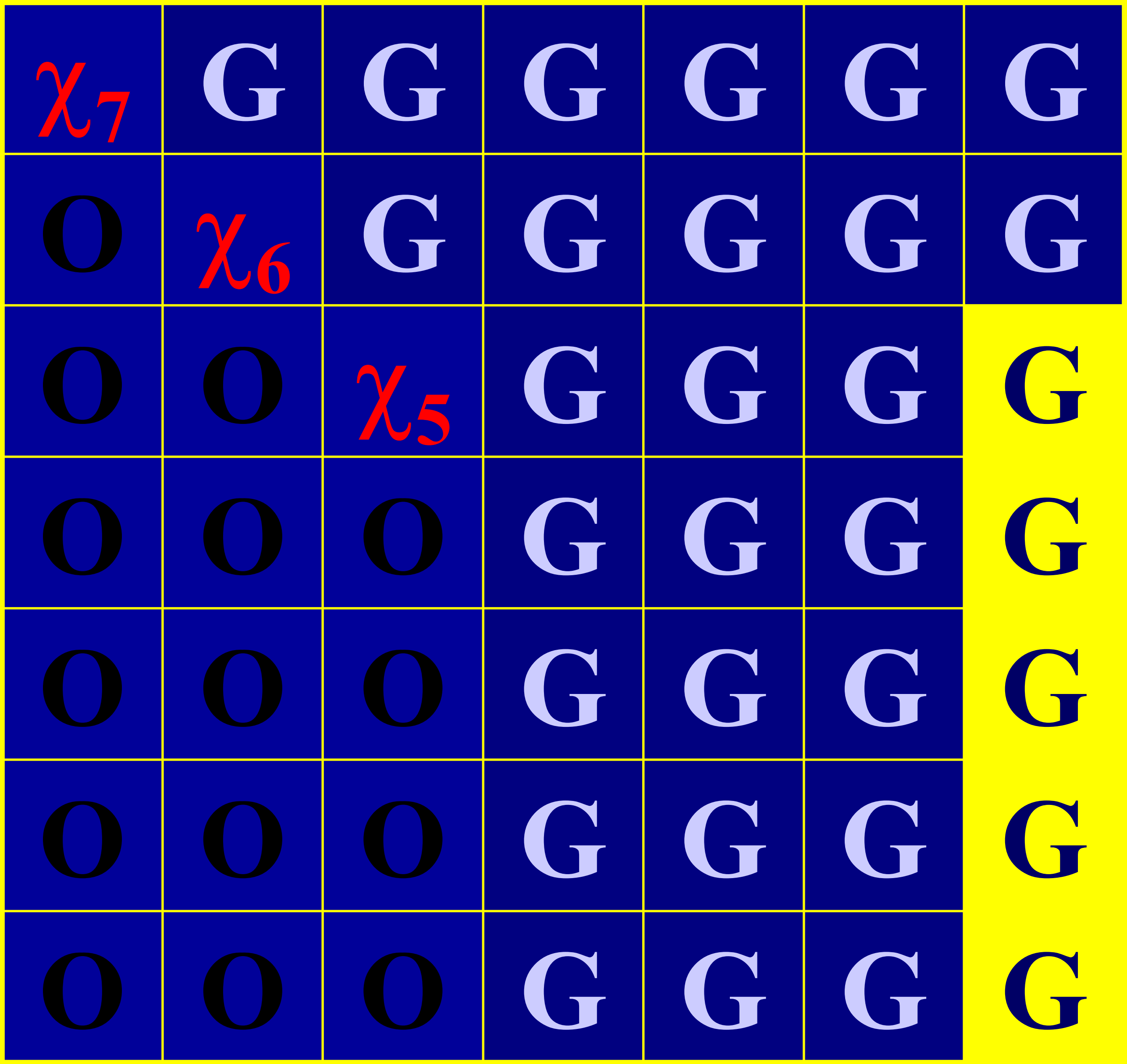
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G

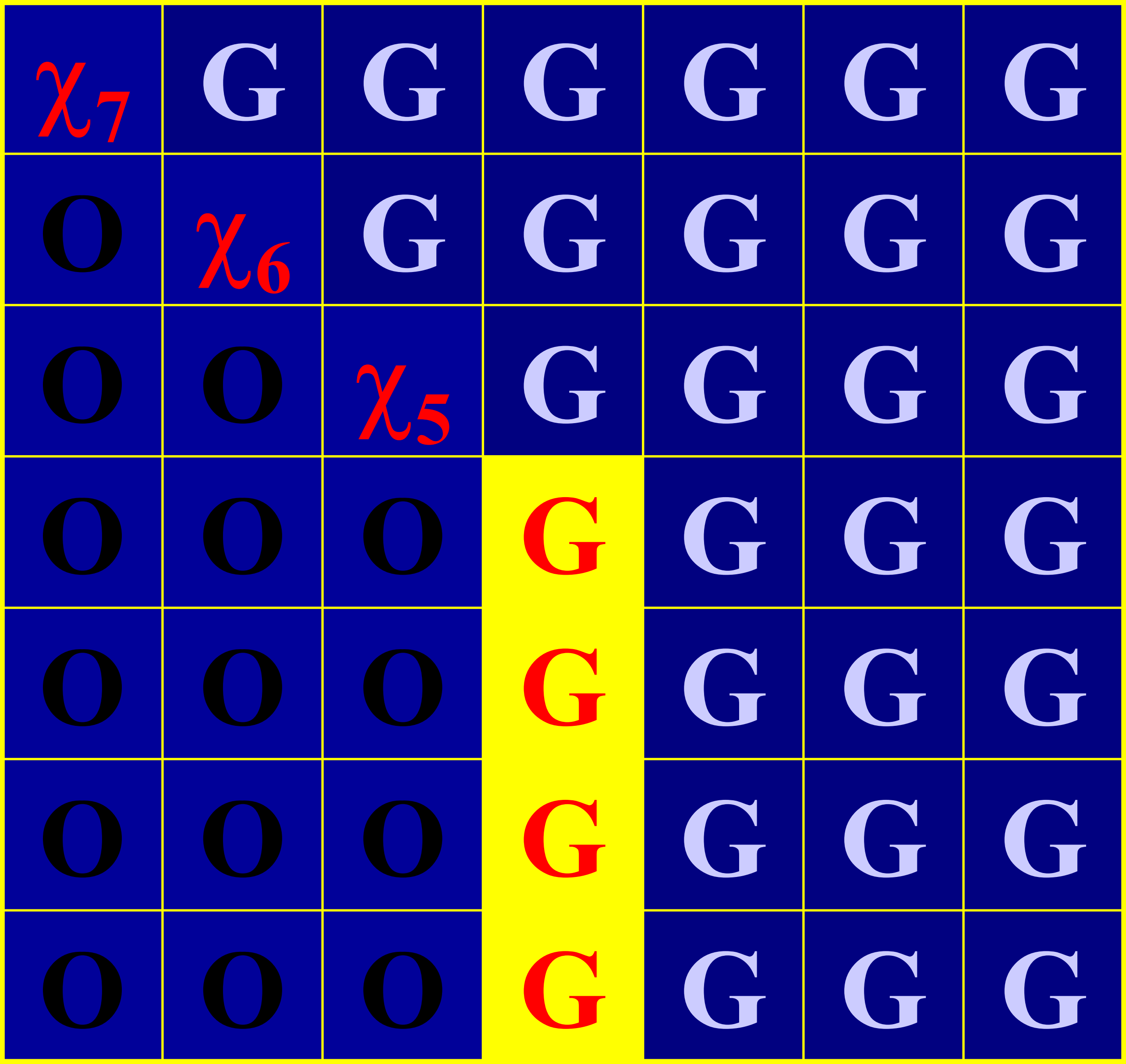


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G

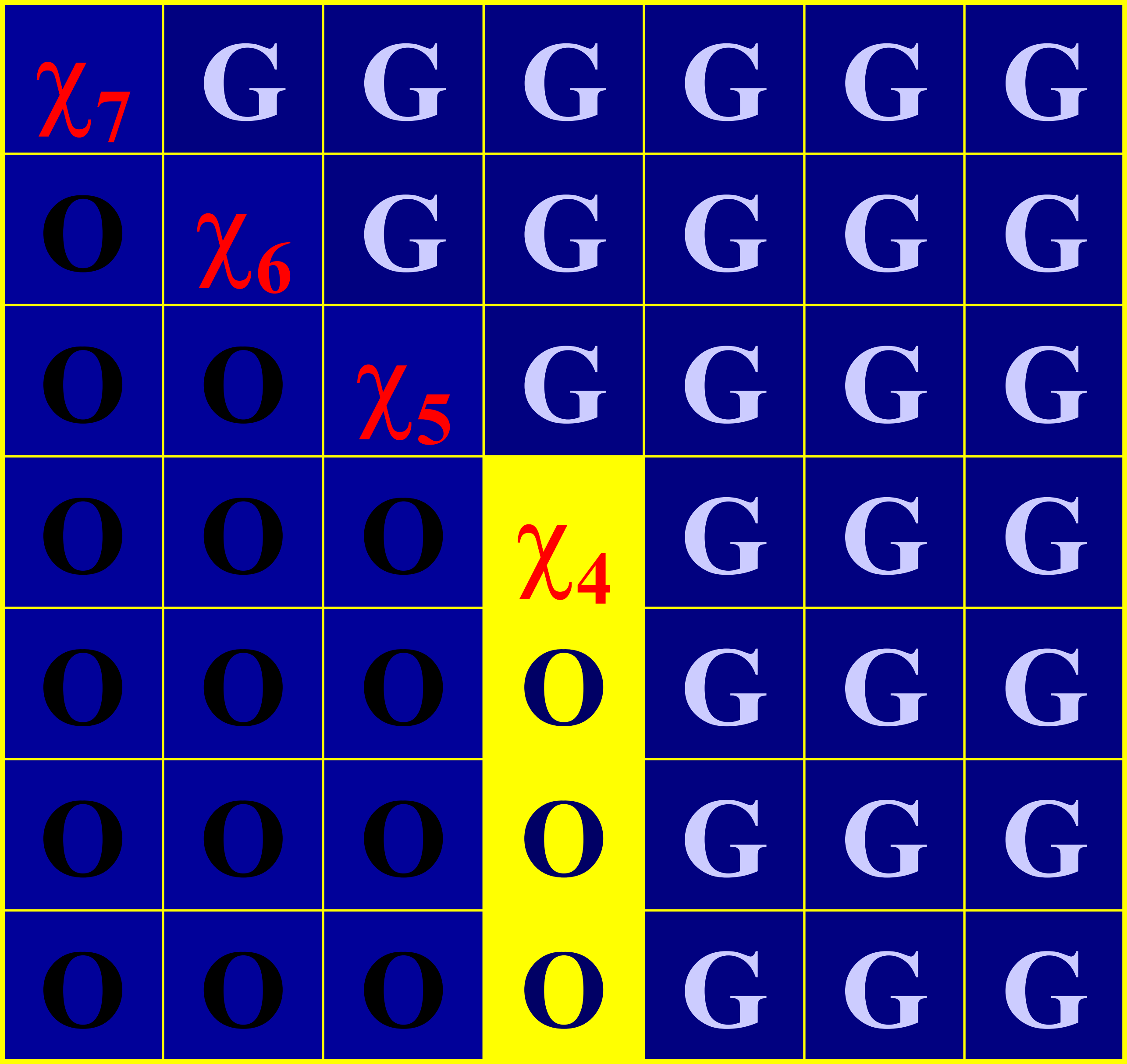


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G

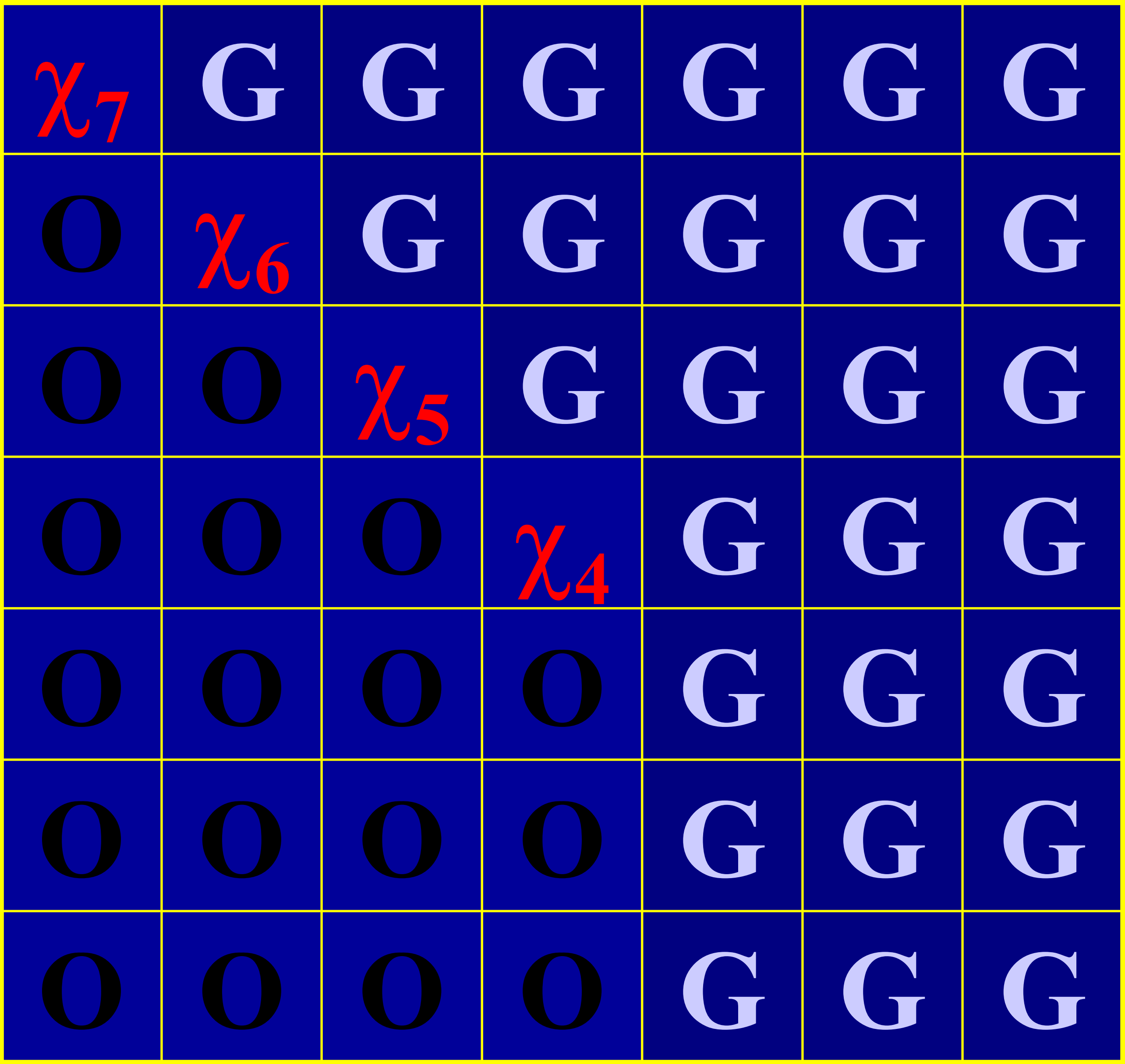
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G
O	O	O	G	G	G	G

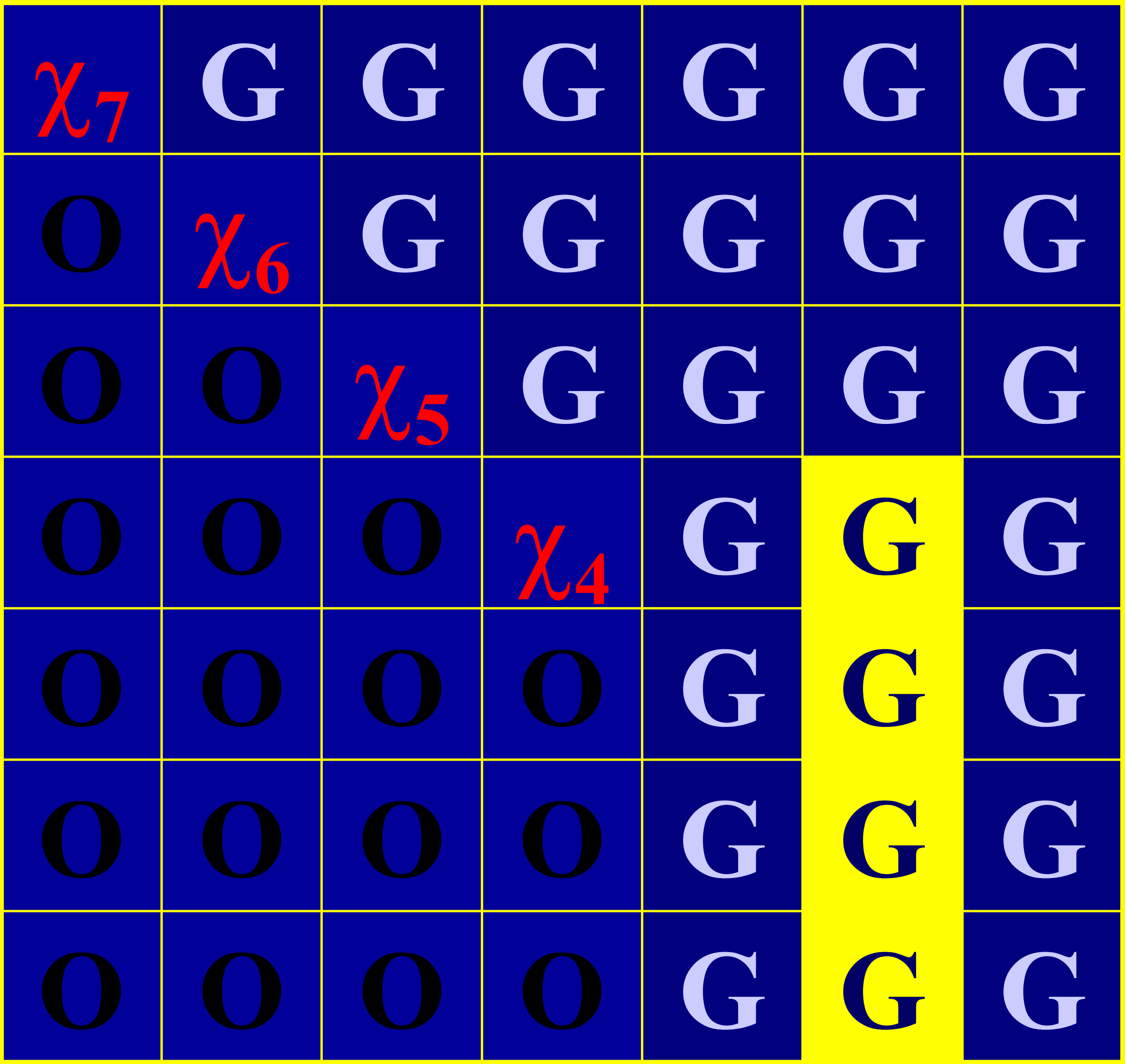


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G

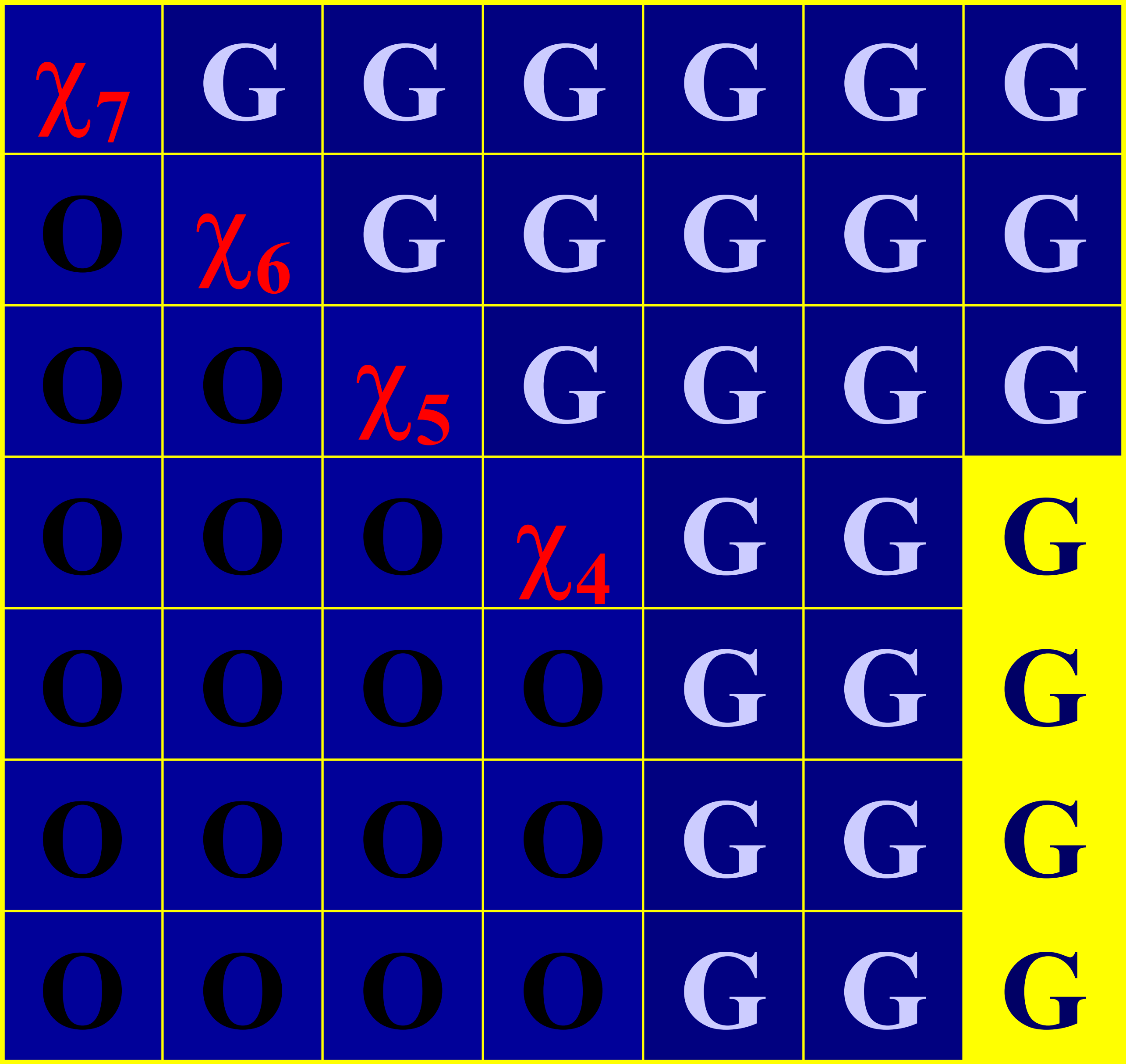


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G

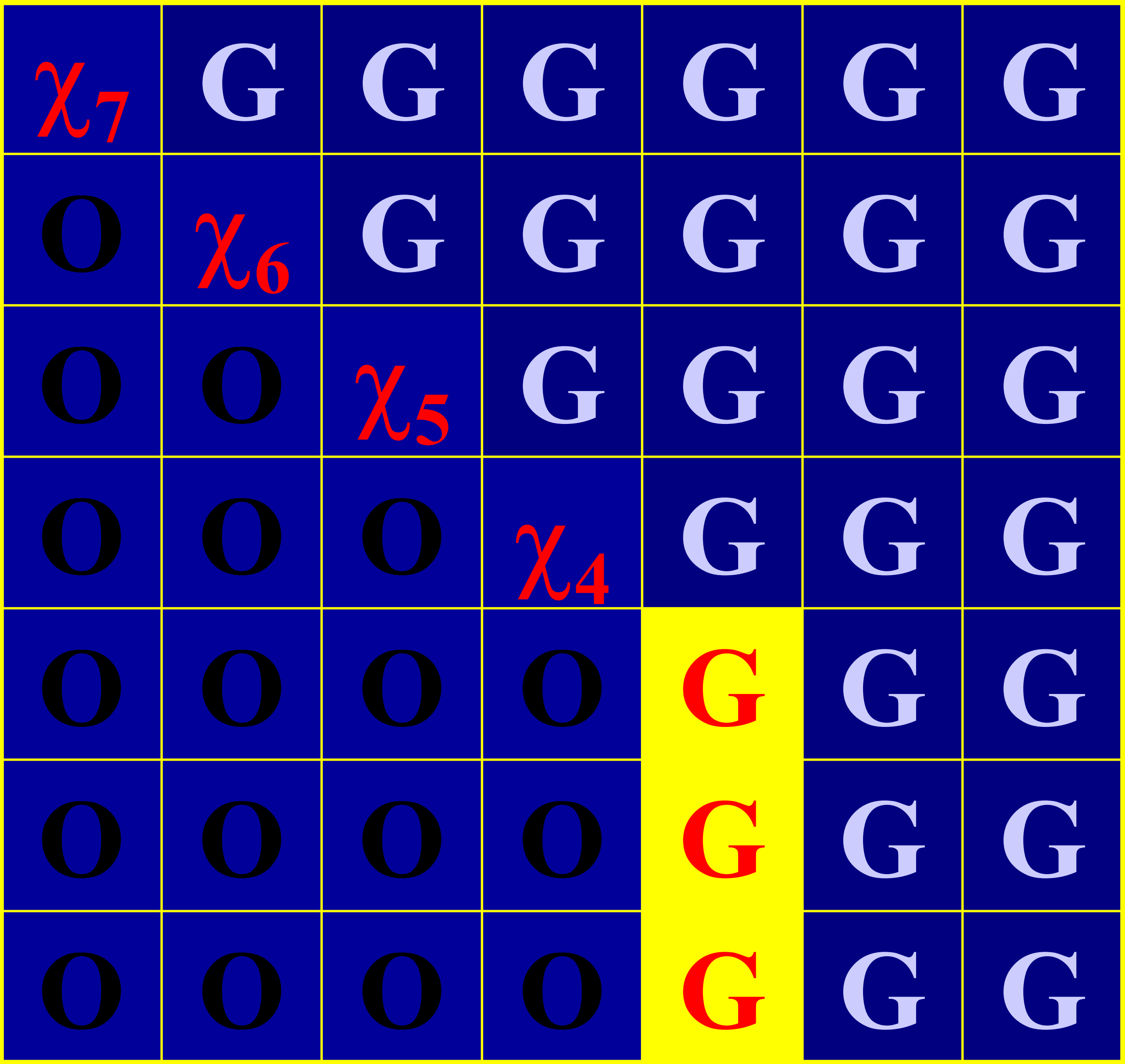


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G

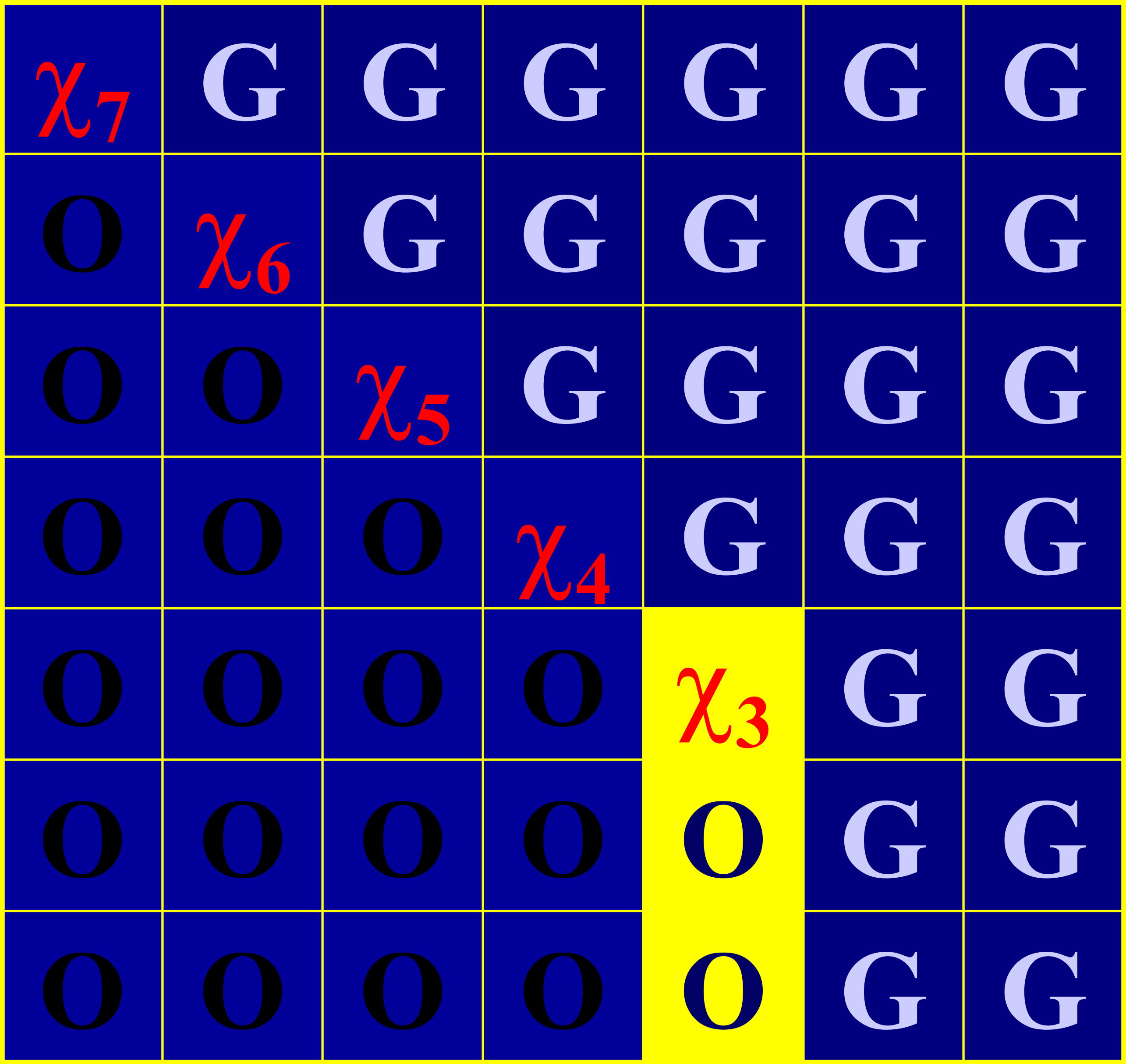


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G



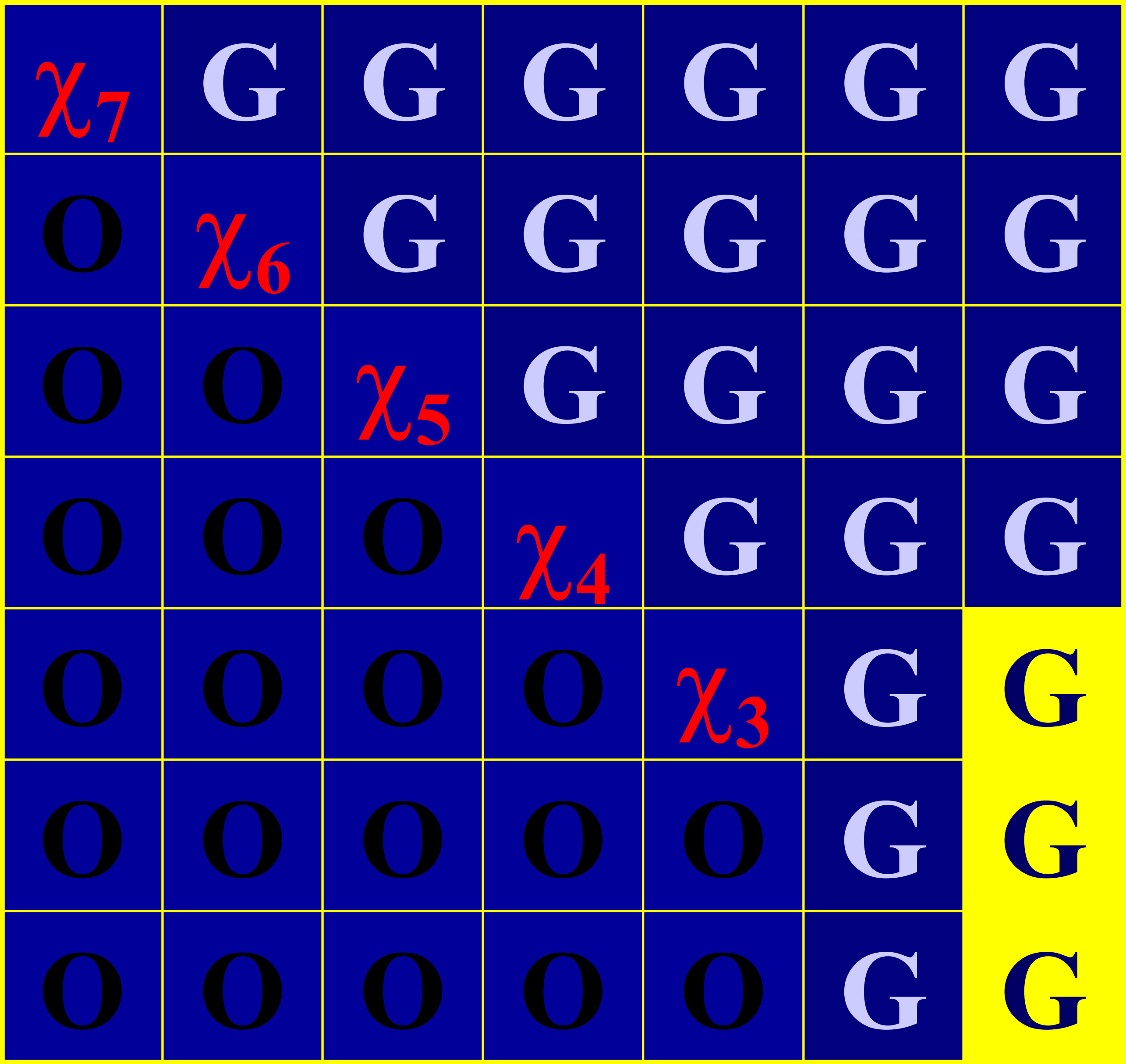
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G
O	O	O	O	G	G	G



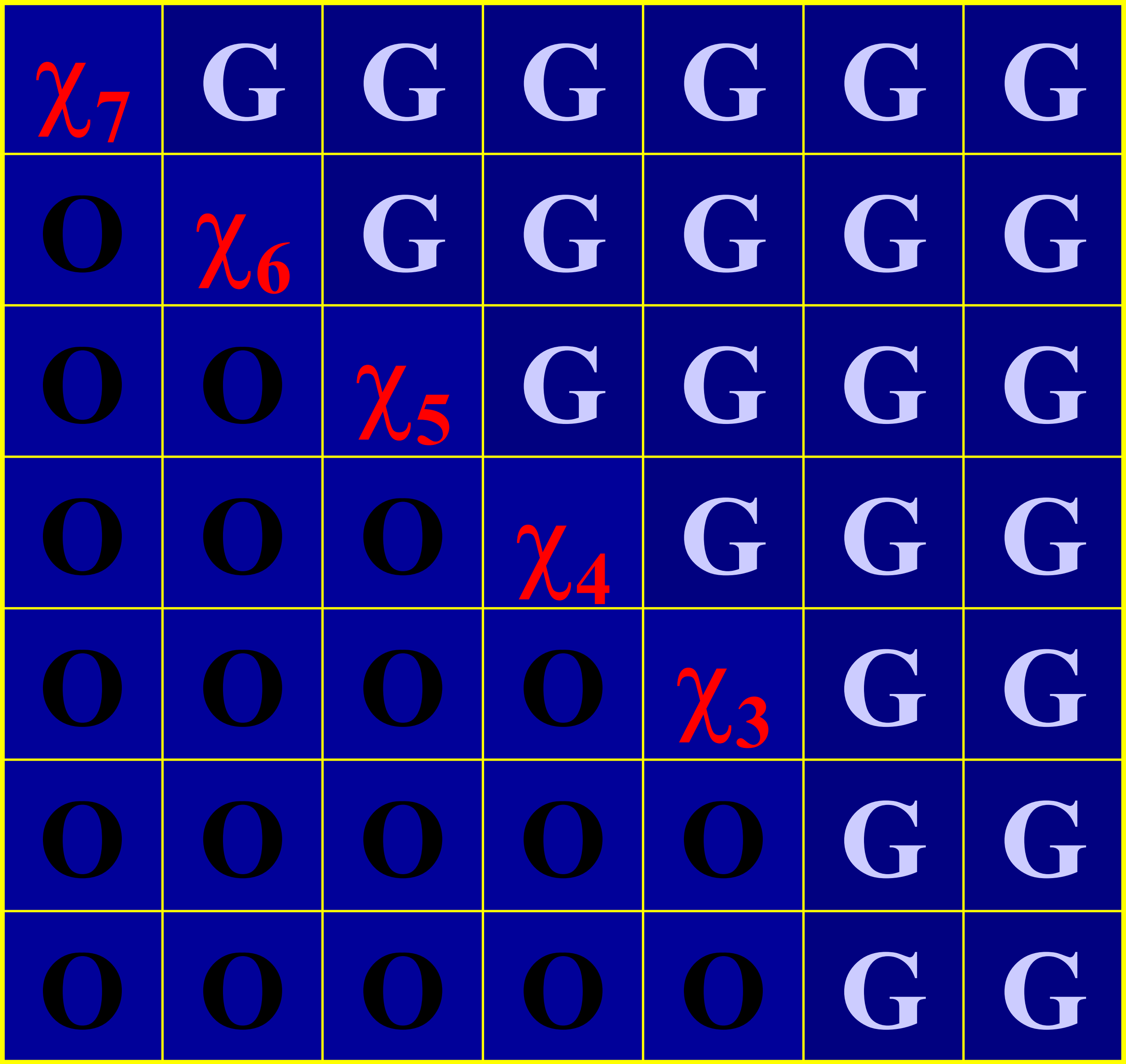
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	G	G
O	O	O	O	O	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	G	G
O	O	O	O	O	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	G	G
O	O	O	O	O	G	G

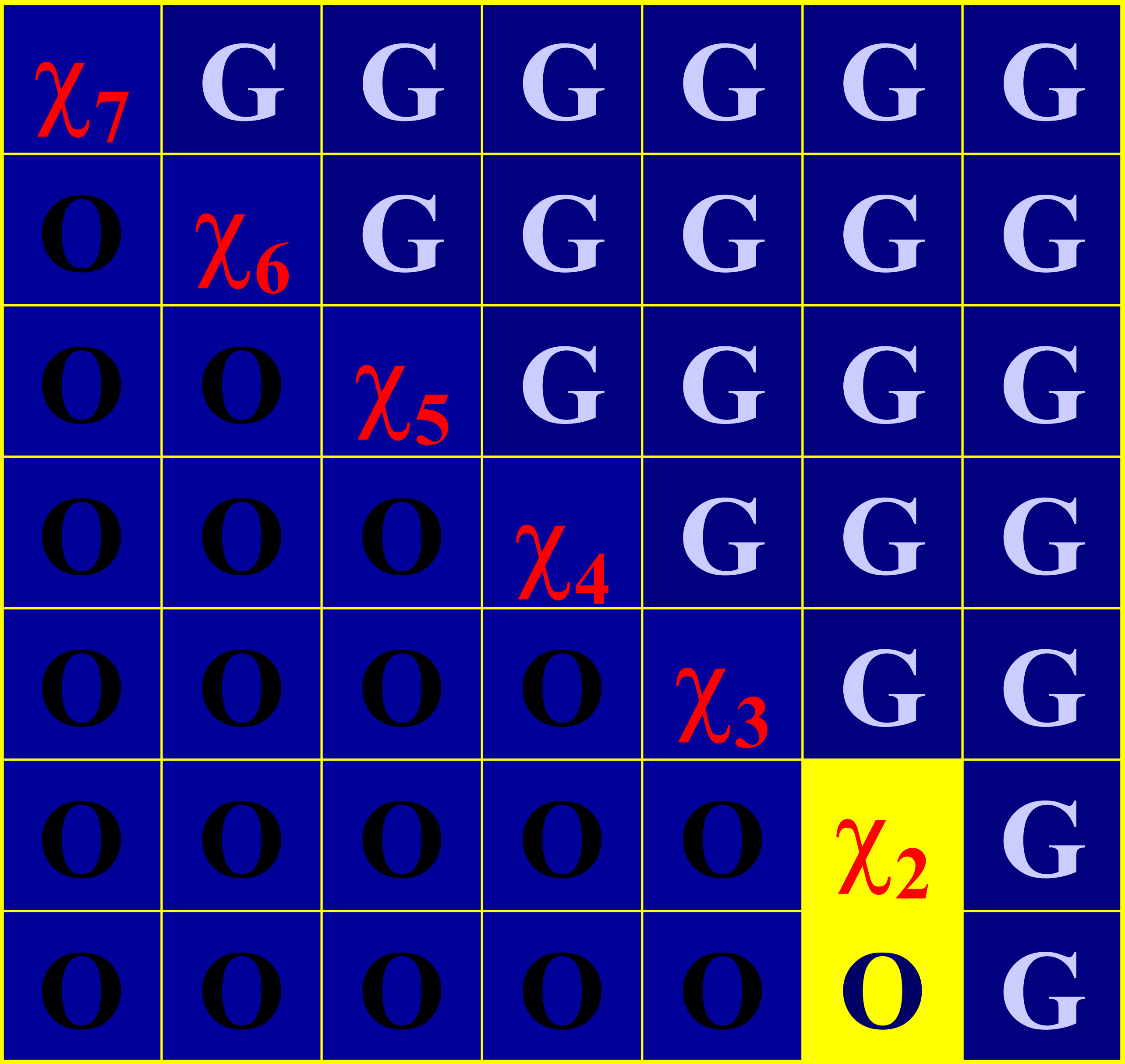


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	G	G
O	O	O	O	O	G	G



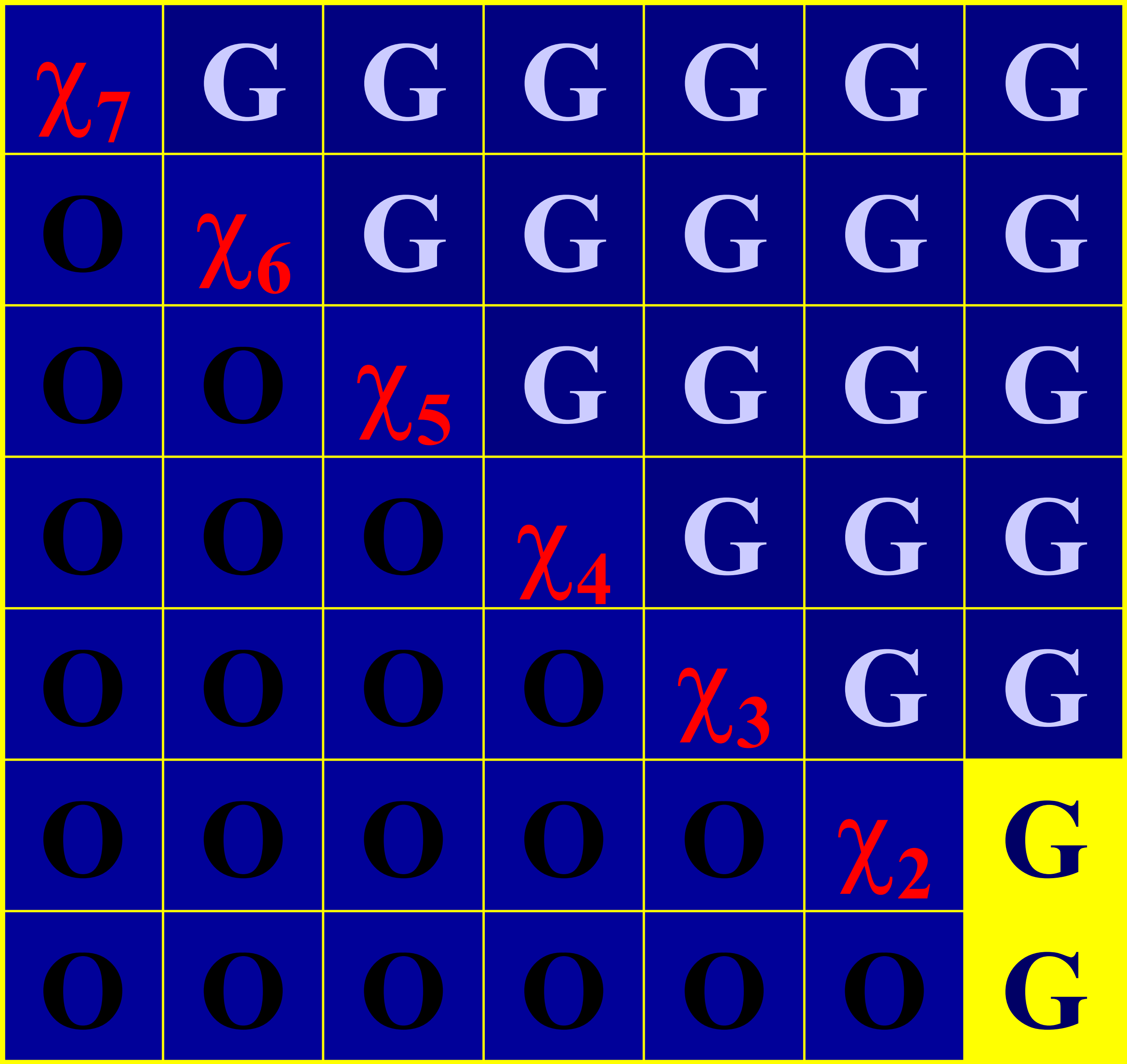
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	G	G
O	O	O	O	O	G	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	G	G
O	O	O	O	O	G	G



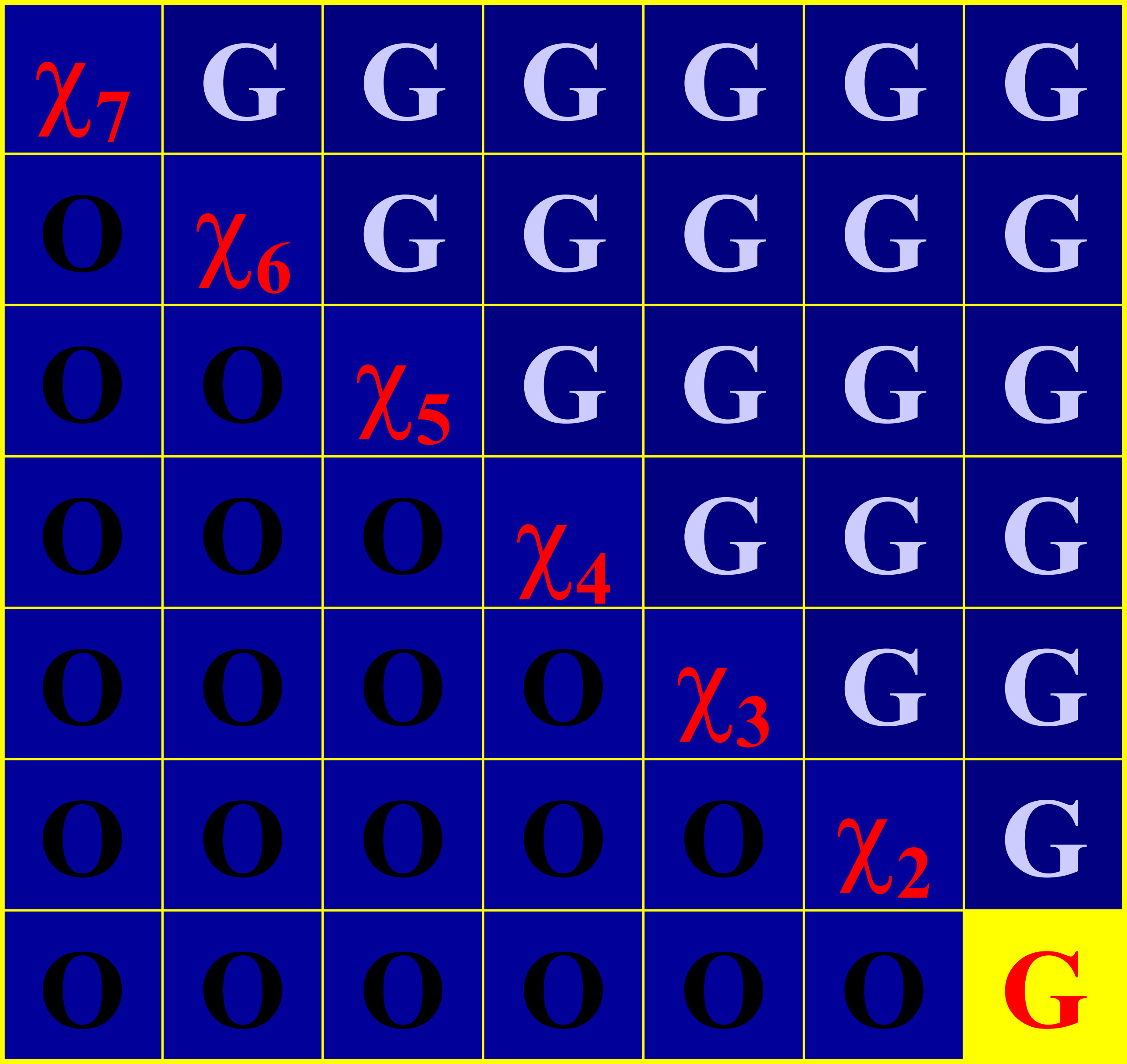
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	G

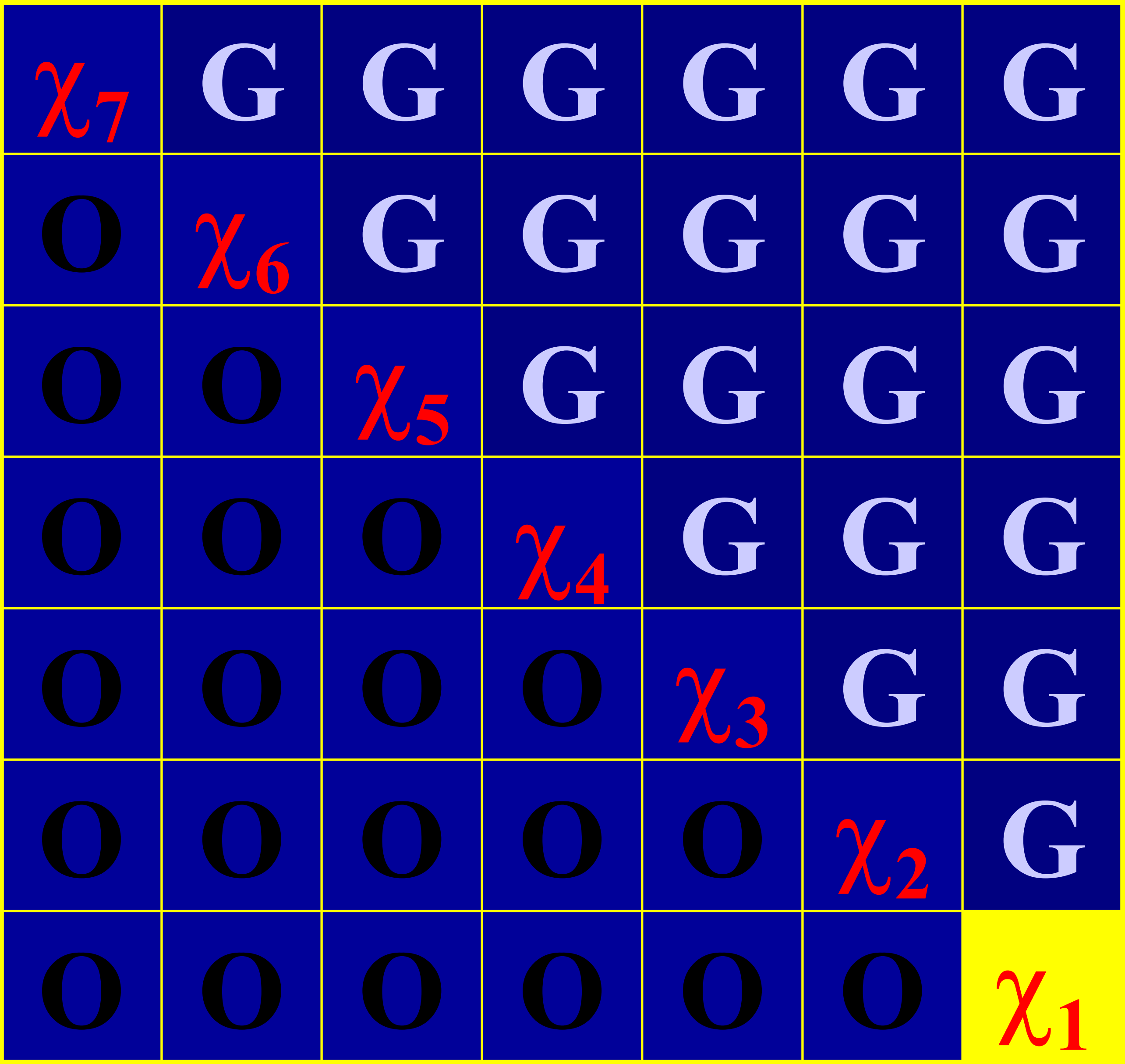


χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	G

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	G



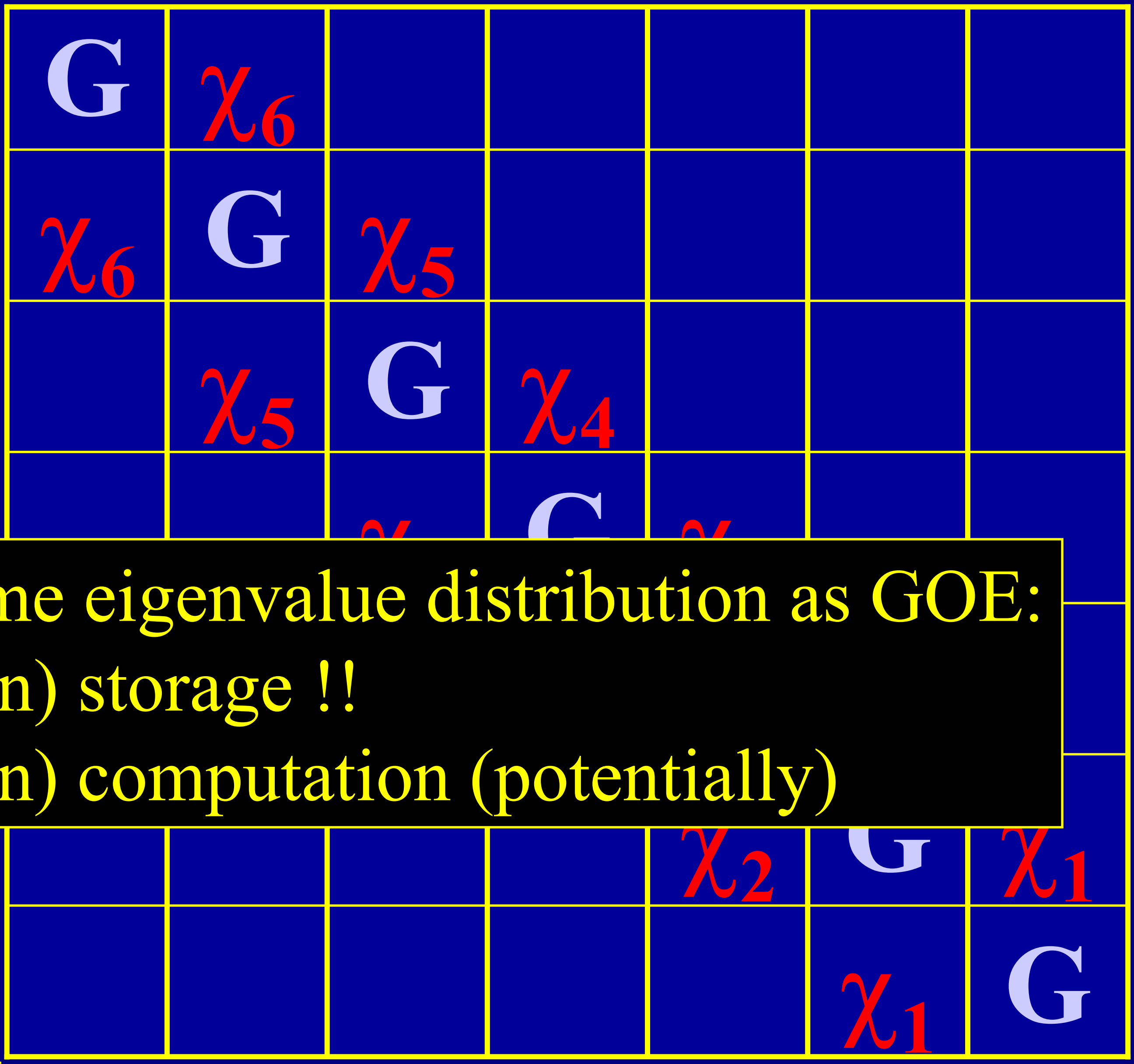
χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	G



χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	χ_1

χ_7	G	G	G	G	G	G
O	χ_6	G	G	G	G	G
O	O	χ_5	G	G	G	G
O	O	O	χ_4	G	G	G
O	O	O	O	χ_3	G	G
O	O	O	O	O	χ_2	G
O	O	O	O	O	O	χ_1

Same idea: sym matrix to tridiagonal form

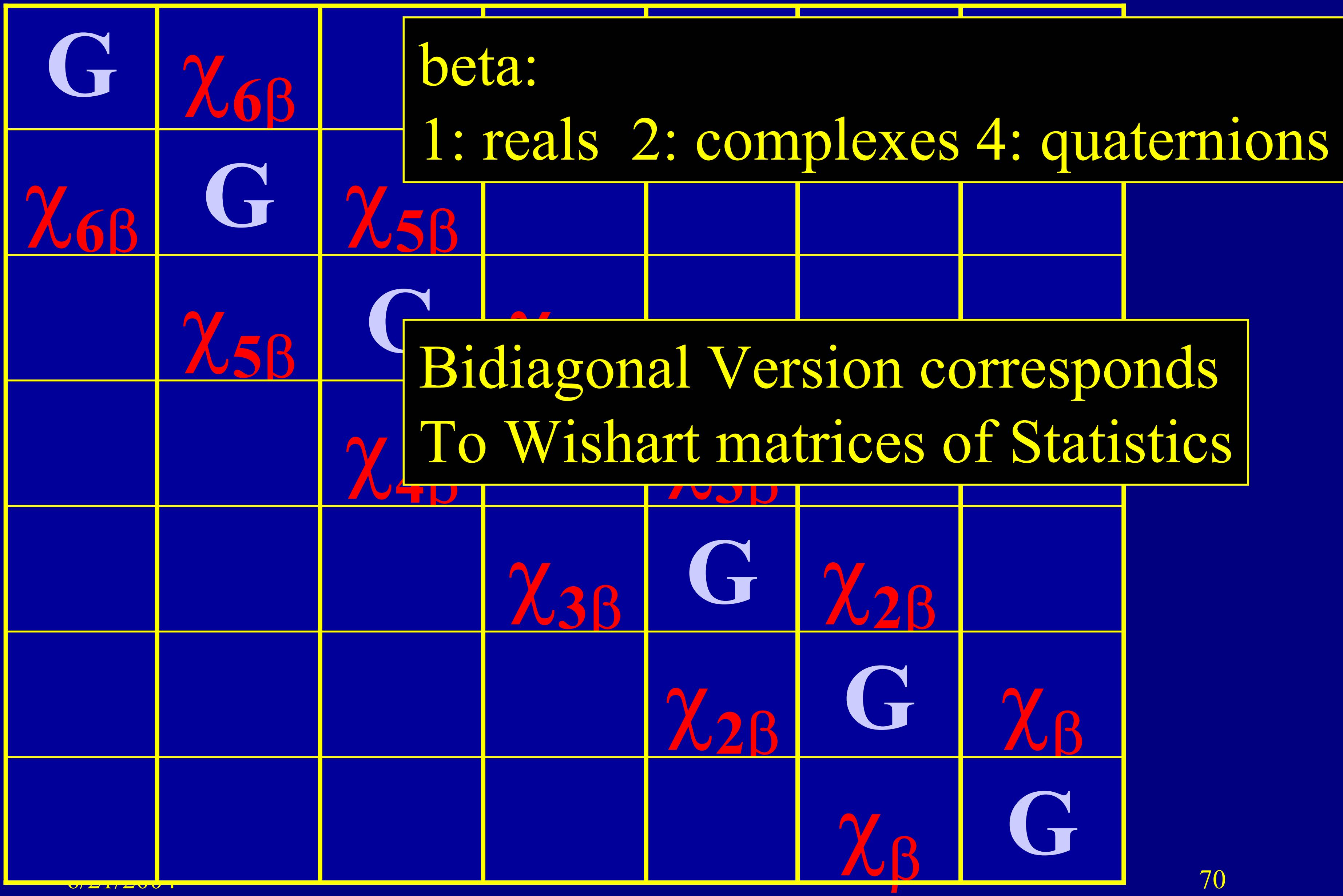


Same eigenvalue distribution as GOE:

O(n) storage !!

O(n) computation (potentially)

Same idea: General beta



G	$\chi_{6\beta}$					
$\chi_{6\beta}$	G	$\chi_{5\beta}$				
	$\chi_{5\beta}$	G	$\chi_{4\beta}$			
		$\chi_{4\beta}$	[illegible]	$\chi_{3\beta}$		
			$\chi_{3\beta}$	G	$\chi_{2\beta}$	
				$\chi_{2\beta}$	G	χ_{β}
					χ_{β}	G

beta:

1: reals 2: complexes 4: quaternions

Bidiagonal Version corresponds
To Wishart matrices of Statistics

Numerical Analysis: Condition Numbers

- $\kappa(A)$ = "condition number of A"
- If $A=U\Sigma V'$ is the svd, then $\kappa(A) = \sigma_{max}/\sigma_{min}$.
- Alternatively, $\kappa(A) = \sqrt{\lambda}_{\ max}\ (A'A)/\sqrt{\lambda}_{\ min}\ (A'A)$
- One number that measures digits lost in finite precision and general matrix "badness"
  - Small=good ☺
  - Large=bad ☹
- The condition of a random matrix???

Von Neumann & co.

- Solve $Ax=b$ via $x= \underbrace{(A'A)^{-1}A'}_{M \approx A^{-1}} b$
- Matrix Residual: $||AM-I||_2$
- $||AM-I||_2 \underset{\approx}{\underset{\uparrow}{<}} 200\kappa^2 n \varepsilon$
- How should we estimate κ?
- Assume, as a model, that the elements of A are independent standard normals!

Von Neumann & co. estimates (1947-1951)

- “For a ‘random matrix’ of order n the expectation value has been shown to be about ~~n~~” ∞

Goldstine, von Neumann

- “… we choose two different values of κ, namely $\boxed{n}$ and $\boxed{\sqrt{10n}}$”

$P(\kappa<n) \approx 0.02$

$P(\kappa< \sqrt{10n}) \approx 0.44$

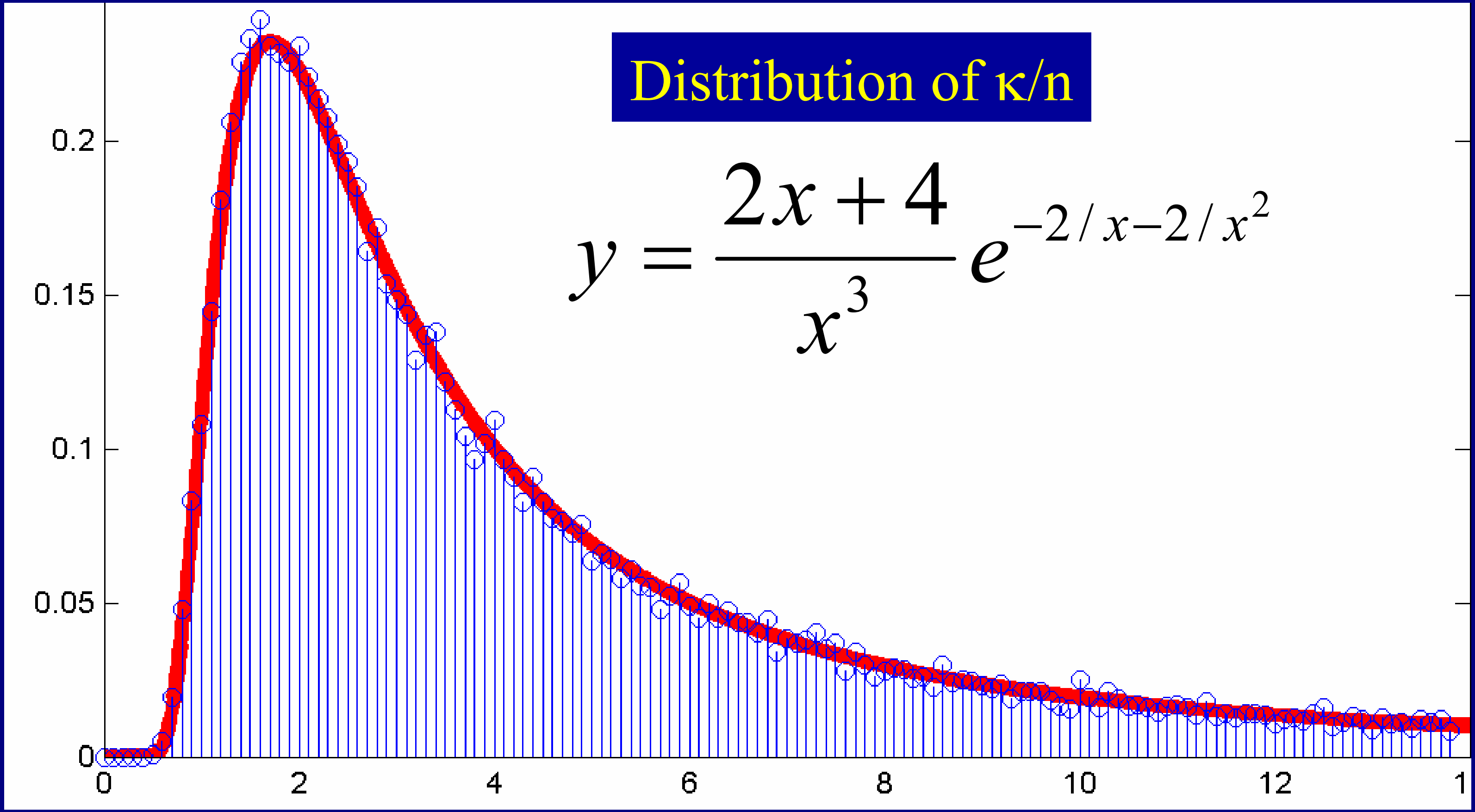
Bargmann, Montgomery, vN

- “With a probability ~1 … $\kappa < \boxed{10n}$”

Goldstine, von Neumann

$P(\kappa<10n) \approx 0.80$

Random cond numbers, n→∞

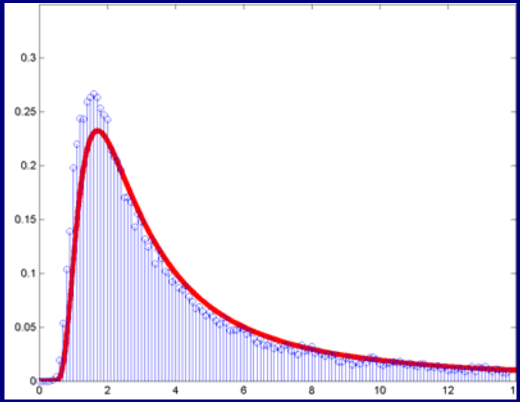


Distribution of κ/n

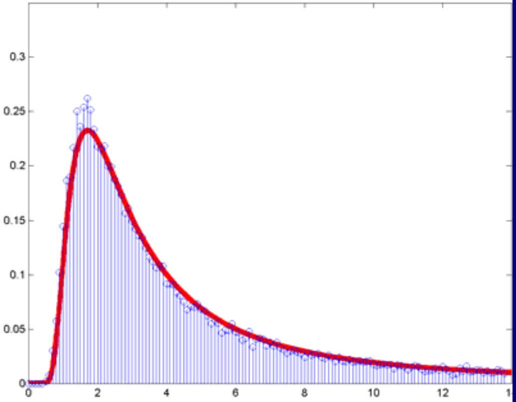
$$y = \frac{2x+4}{x^3} e^{-2/x-2/x^2}$$

Experiment with n=200

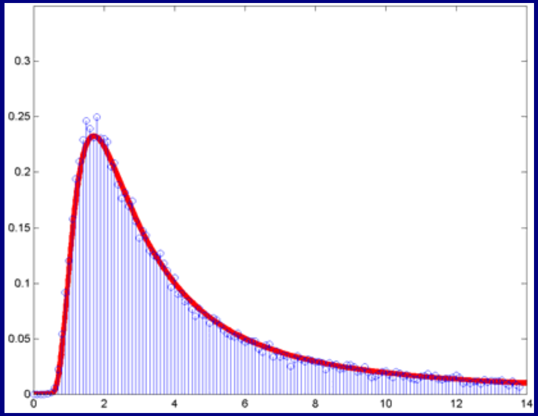
Finite n



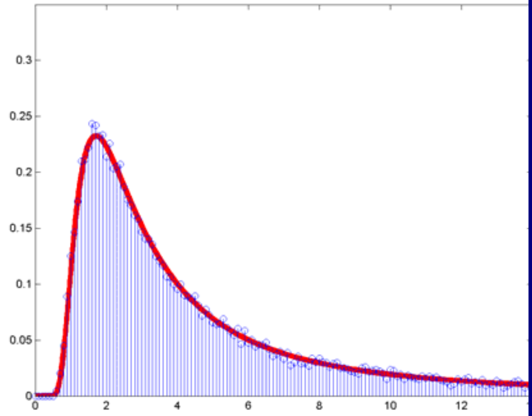
n=10



n=25



n=50



n=100

Small World Networks:

& 6 degrees of separation

- Edelman, Eriksson, Strang
- Eigenvalues of A=T+PTP’, P=randperm(n)

$$T = \begin{bmatrix} 0 & 1 & & & & & & 1 \\ 1 & 0 & 1 & & & & & \\ & 1 & 0 & 1 & & & & \\ & & 1 & 0 & \ddots & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & 1 & 0 & 1 & \\ & & & & & 1 & 0 & 1 \\ 1 & & & & & & 1 & 0 \end{bmatrix}$$

Incidence matrix of graph with two superimposed cycles.

Small World Networks:

& 6 degrees of separation

- Edelman, Eriksson, Strang
- Eigenvalues of A=T+PTP’, P=randperm(n)

$$T = \begin{bmatrix} 0 & 1 & & & & & & 1 \\ 1 & 0 & 1 & & & & & \\ & 1 & 0 & 1 & & & & \\ & & 1 & 0 & \ddots & & & \\ & & & \ddots & \ddots & 1 & & \\ & & & & 1 & 0 & 1 & \\ & & & & & 1 & 0 & 1 \\ 1 & & & & & & 1 & 0 \end{bmatrix}$$

Incidence matrix of graph with two superimposed cycles.

- Wigner style derivation counts number of paths on a tree starting and ending at the same point (tree = no accidents!) (McKay)
- We first discovered the formula using the superseeker
- Catalan number answer d^{2n-1}-Σ $d^{2j-1}(d-1)^{n-j+1}C_{n-j}$

The Riemann Zeta Function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1} du = \sum_{k=1}^\infty \frac{1}{k^s}$$

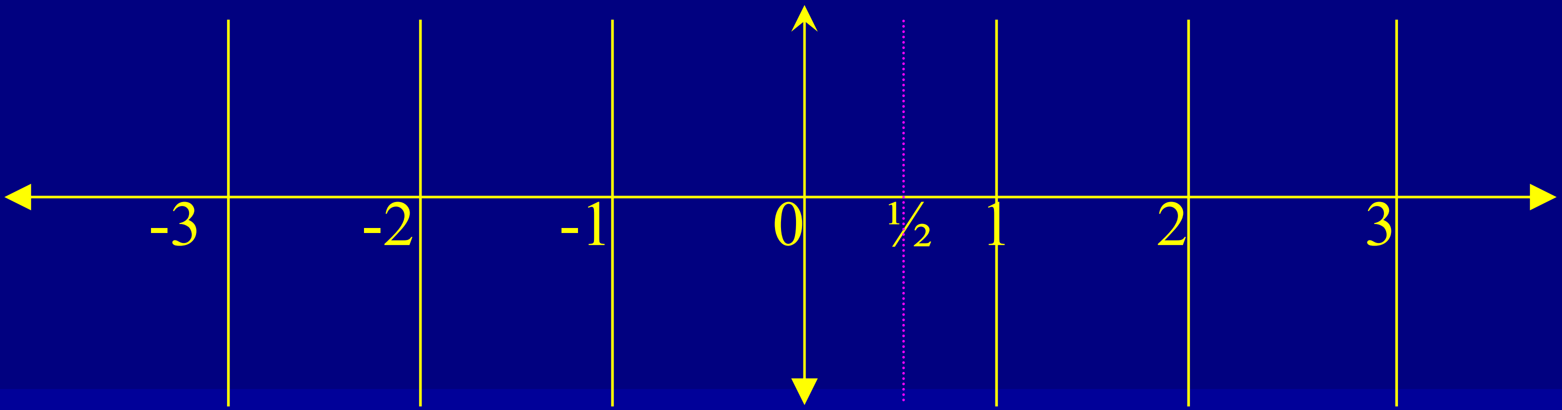
On the real line with x>1, for example

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$$

May be analytically extended to the complex plane, with singularity only at x=1.

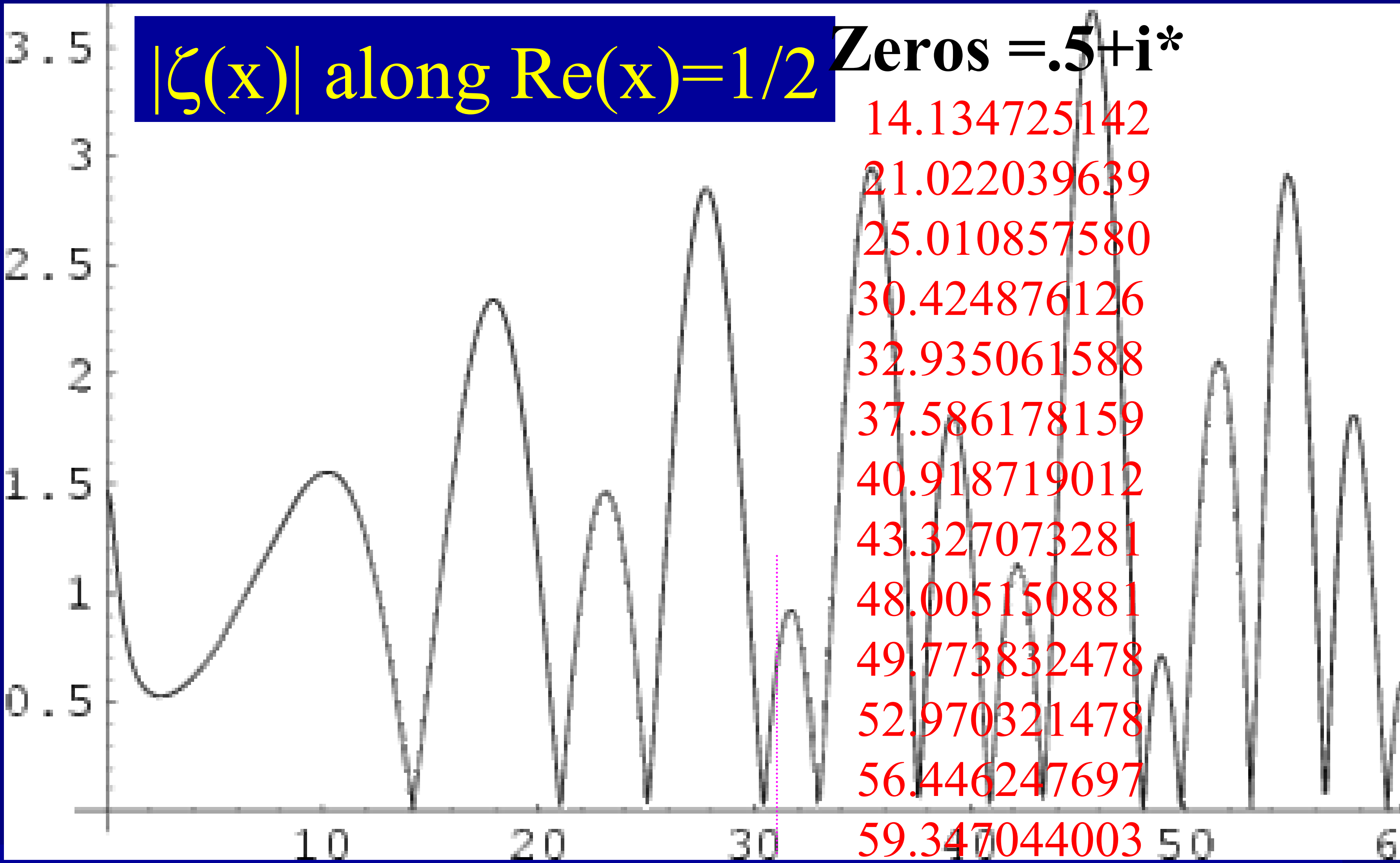
The Riemann Hypothesis

$$\zeta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{u^{x-1}}{e^u - 1}du = \sum_{k=1}^\infty \frac{1}{k^x}$$



All nontrivial roots of ζ(x) satisfy Re(x)=1/2.
(Trivial roots at negative even integers.)

$|\zeta(x)|$ along $\mathrm{Re}(x)=1/2$



Zeros =.5+i*

14.134725142
21.022039639
25.010857580
30.424876126
32.935061588
37.586178159
40.918719012
43.327073281
48.005150881
49.773832478
52.970321478
56.446247697
59.347044003

All nontrivial roots of $\zeta(x)$ satisfy $\mathrm{Re}(x)=1/2$.
(Trivial roots at negative even integers.)

Computation of Zeros

- Odlyzko's fantastic computation of 10^k+1 through 10^k+10,000 for k=12,21,22.

See http://www.research.att.com/~amo/zeta_tables/

Spacings behave like the eigenvalues of

A=randn(n)+i*randn(n); S=(A+A')/2;

Nearest Neighbor Spacings & Pairwise Correlation Functions

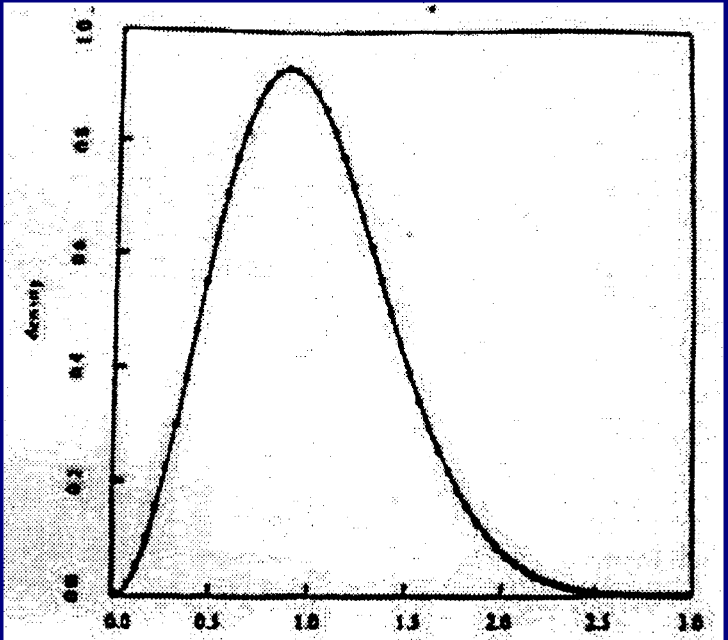


FIGURE 1
Nearest neighbor spacings among 70 million zeros beyond the 10^{20}-th zero of zeta, versus $\mu_1(GUE)$

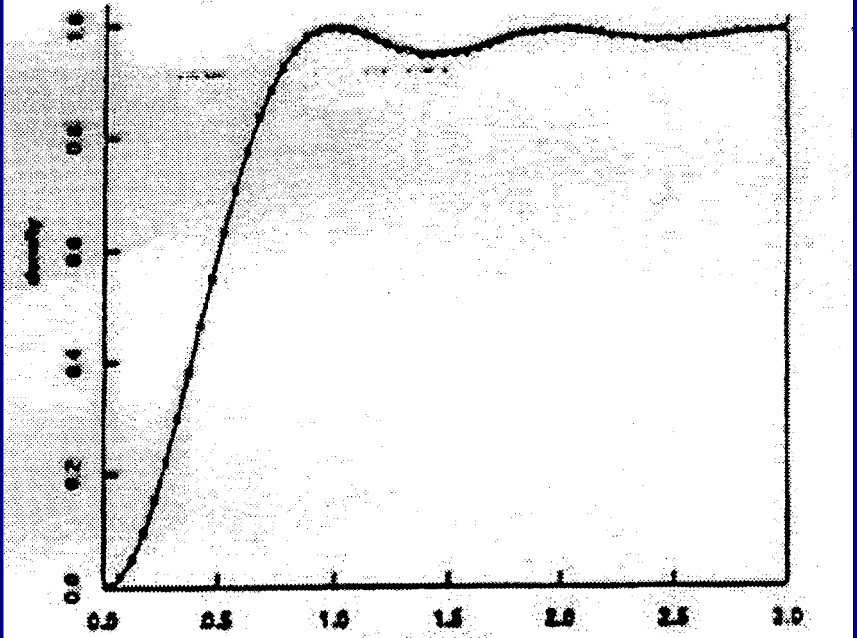


FIGURE 2
Pair correlation for zeros of zeta based on 8×10^6 zeros near the 10^{20}-th zero, versus the GUE conjectured density $1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$.

Painlevé Equations

$$\begin{aligned}
\text{I)} \qquad y'' &= 6y^2 + t, \\
\text{II)} \qquad y'' &= 2y^3 + ty + \alpha, \\
\text{III)} \qquad y'' &= \frac{1}{y}y'^2 - \frac{y'}{t} + \frac{\alpha y^2 + \beta}{t} + \gamma y^3 + \frac{\delta}{y}, \\
\text{IV)} \qquad y'' &= \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{V)} \qquad y'' &= \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t}\left(\alpha y + \frac{\beta}{y}\right) \\
&\qquad + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}, \\
\text{VI)} \qquad y'' &= \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
&+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha - \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{t(t-1)}{(y-t)^2}\right]
\end{aligned}$$

Spacings

- Take a large collection of consecutive zeros/eigenvalues.
- Normalize so that average spacing = 1.
- Spacing Function = Histogram of consecutive differences (the (k+1)st – the kth)
- Pairwise Correlation Function = Histogram of all possible differences (the kth – the jth)
- Conjecture: These functions are the same for random matrices and Riemann zeta

Some fun tidbits

- The circular law
- The semi-circular law
- Infinite vs finite
- How many are real?
- Stochastic Numerical Algorithms
- Condition Numbers
- Small networks
- Riemann Zeta Function
- Matrix Jacobians

Matrix Factorization Jacobians

General

$A=LU$ $\prod u_{ii}^{n-i}$ $A=QR$ $\prod r_{ii}^{m-i}$

$A=U\Sigma V^T$ $\prod (\sigma_i^2-\sigma_j^2)$ $A=QS$ (polar) $\prod (\sigma_i+\sigma_j)$

$A=X\Lambda X^{-1}$ $\prod (\lambda_i-\lambda_j)^2$

Sym

$S=Q\Lambda Q^T$ $\prod (\lambda_i-\lambda_j)$

$S=LL^T$ $2^n\prod l_{ii}^{n+1-i}$

$S=LDL^T$ $\prod d_i^{n-i}$

Orthogonal

$Q=U\begin{bmatrix} c & s \\ s & -c \end{bmatrix}V^T$ $\prod \sin(\theta_i+\theta_j)\sin(\theta_i-\theta_j)$

Tridiagonal

$T=Q\Lambda Q^T$ $\prod (t_{i+1,i})/\prod q_i$

Why cool?

- Why is numerical linear algebra cool?
  - Mixture of theory and applications
  - Touches many topics
  - Easy to jump in to, but can spend a lifetime studying & researching
- Tons of activity in many areas
  - Mathematics: Combinatorics, Harmonic Analysis, Integral Equations, Probability, Number Theory
  - Applied Math: Chaotic Systems, Statistical Mechanics, Communications Theory, Radar Tracking, Nuclear Physics
- Applications
- BIG HUGE SUBJECT!!