Lovely Pairs

Pair: (M, N) where $M \succ N$ (F.O.)

another way of writing it: (M, P) where P is a new unary predicate & $P(M) = N$.

We even allow (A, P) where $A \subseteq M$ (some model) and $P(A)$ is relatively algebraically closed in A.

Def: Fix a simple theory T.

A pair (M, P) is κ-lovely where $\kappa > |T|$ if

~~...~~ $\forall A \subseteq M$ s.t. $|A| < \kappa$ and $\forall p \in S(A)$ (in the sense of T)

① $\exists a \in M$ $a \models p$ and $a \underset{A}{\perp} P(M)$

② If moreover p dnd $/ P(A)$ then $\exists a' \in P(M)$ $a' \models p$.

Lovely is $|T|^+$-lovely.

We call ① the extension property: every $p \in S(A)$ has a nondividing extension to $A \cup P(M)$ realised in M.

We call ② the co-heir property: it says if $p \in S(M)$ which does not divide $/P(M)$ then every small part of p is realised in $P(M)$.

Note: if (M, P) is κ-lovely then both $\overset{\text{from ①}}{M}$, $\overset{\text{from ②}}{P(M)}$ are κ-saturated models of T.

Fact: κ-lovely pairs exist for arbitrarily big κ.

Defn: Let (M, P) be a pair and $A \subseteq M$.

A is *free* if $A \underset{P(A)}{\downarrow} P(M)$.

An embedding of pairs is free if it respects P, and the image is free.
both ways

Lemma Assume (M, P), (N, P) are lovely pairs of T,

$A \subseteq M$ is free, $|A| \leq |T|$,

$B \subseteq N$ is free, $|B| \leq |T|$, and

$\exists f : A \xrightarrow{\sim} B$ preserving T-types and P.

Then $\forall c \in M \ \exists A' \supseteq Ac$, $B' \supseteq B$ st. same holds for A', B' via f' extending f.

(Back & forth but not f.o. because of freeness)

Proof: Case I: $c \in P(M)$.

Then $A \underset{P(A)}{\downarrow} c$. Define $A' = Ac$, $P(A') = P(A), c$.

So A' is free.

Find $d \in N$ st. $dB \equiv cA$. Then $d \underset{P(B)}{\downarrow} B$ so we may choose $d \in P(N)$ by coheir prop.

Case II: $c \notin P(M) = \mathrm{bdd}(P(M))$. ($T^+$-sat model so bdd-closed)

Find $G \subseteq P(M)$ s.t. $|G| \leq |T|$ and $Ac \underset{G}{\downarrow} P(M)$ (local char.)

WMA $G \supseteq P(A)$.

Let $A' := AGc$ so $P(A') = G$

By case I, find $D \subseteq P(N)$ and $f' : AG \to BD$

Find $d \in M$ st. ~~...~~ $AGc \underset{\text{in sense of } T}{\overset{\text{via } f''}{\equiv}} BDd$ $(f'' \supseteq f' \supseteq f)$

We may choose it st. $d \underset{BD}{\downarrow} P(N)$ by extn prop.

Since $P(AG) = G$, we have $P(BD) = D$.

So $B \underset{D}{\downarrow} P(N) \Rightarrow Bd \underset{D}{\downarrow} P(N)$

Set $B' = BDd$: $B' \underset{P(B')}{\downarrow} P(N)$

Left to prove $d \notin P(N)$: if $d \in P(N)$ then $d \underset{D}{\downarrow} d$

$\Rightarrow d \in \mathrm{bdd}(D) \Rightarrow c \in \mathrm{bdd}(CG) \subseteq P(M)$ contradiction. $\square$

For now, assume T is a complete f.o. simple theory with QE.

Let $\mathcal{L}_P = \mathcal{L} \cup \{P\}$. Then: if (M,P) and (N,P) are lovely T-pairs then $Th_{\mathcal{L}_P}(M,P) = Th_{\mathcal{L}_P}(N,P)$:

start a back and forth between (M,P) and (N,P) from $\emptyset \mapsto \emptyset$.

Moreover: if $A \subseteq (M,P)$ is free then $tp^{(M,P)}_{\mathcal{L}_P}(A)$ is determined by $tp^{M}_{\mathcal{L}}(A)$ and the trace of P on A.

Define $T_P := Th_{\mathcal{L}_P}(\text{lovely pairs})$. T_P is complete.

<u>Lemma</u> Let (A,P) be a pair. Then it embeds freely in a κ-lovely pair $(\forall \kappa)$.

<u>Proof</u> Let (M,P) be a ~~κ-lovely pair~~ $(\kappa + |A|^{+})$-lovely pair.

First embed $P(A)$ in $P(M)$.

Realise $tp(A/P(A))$ in M s.t. $A \underset{P(A)}{\perp} P(M)$ $\square$

Every model of T_P is a pair, and therefore can be embedded freely in a lovely pair.

Moreover, it is easy to verify: if $(M,P) \preceq (N,P) \models T_P$ then M is free in (N,P).

Converse? True if $(M,P),(N,P)$ are lovely.

(If $(M,P) \overset{\text{free}}{\hookrightarrow} (N,P)$ and $A \subseteq M$ is free then it is free in N.

$M \underset{P(M)}{\perp} P(N) \Rightarrow A \underset{P(M)}{\perp} P(N) \Rightarrow A \underset{P(A)}{\perp} P(N)$)

The Big Theorem: TFAE (for T):

1. Every free extension of models of T_P is elementary
2. Every model of T_P embeds elementarily in a lovely pair.
3. Every κ-lovely pair is κ-saturated as an $\mathcal{L}_P$-structure.
4. There exists a lovely pair that is $|T|^+$-saturated as an $\mathcal{L}_P$-structure.

Proof ① $\Rightarrow$ ②: Since every pair embeds ~~free~~ freely in a lovely pair & assumption.

② $\Rightarrow$ ③ Let (M,P) be κ-lovely, let $A \subseteq M$ $|A| < \kappa$.

Let $(N,P) \succcurlyeq (M,P)$ $a \in N$. We want to show $\exists a' \in M$

s.t. $a \equiv_A^{\mathcal{L}_P} a'$.

By ② we may assume that (N,P) is a lovely pair. (replace by elt extn).

~~Enlarging A but keeping~~

Find $C \subseteq P(N)$ st. $aA \underset{C}{\perp} P(N)$ and $|C| < \kappa$ $(\kappa > |T|)$

Enlarging A but keeping $|A| < \kappa$, we may assume A is free in M and therefore in N (same arg as for C).

Now we get $A \underset{P(A)}{\downarrow} P(N) \Rightarrow A \underset{P(A)}{\downarrow} C$ (coheir)

$\Rightarrow \exists\, C' \subseteq P(M)$ s.t. $C' \underset{A}{\equiv}^{\mathcal{L}} C$.

Now $\exists\, a' \in M$ s.t. $aC \underset{A}{\equiv} a'C'$ and $a' \underset{AC'}{\downarrow} P(M)$ (extn)

Then $P(aAC) = P(A)C$ and $P(a'AC') = P(A)C'$
(so traces are the same)

and $aAC \underset{P(A)C}{\downarrow} P(N)$ (since $= aA \underset{P(A)}{\downarrow} P(N)$)

and $A \underset{P(A)C'}{\downarrow} P(M)$ and $\Rightarrow a'AC' \underset{P(A)C'}{\downarrow} P(M)$.

ie ACa is free in N and $AC'a'$ is free in M.

$\Rightarrow \quad ACa \equiv^{\mathcal{L}_P} AC'a' \Rightarrow a' \underset{A}{\equiv}^{\mathcal{L}_P} a'$

③ $\Rightarrow$ ④ by existence.

④ $\Rightarrow$ ① next time.

5/12. <u>From last time</u>: (M,P) and $A \subseteq M$ then A is free if $A \underset{P(A)}{\downarrow} P(M)$

If (M,P) lovely and $A \subseteq M$ is free then $tp^{\mathcal{L}}(A) + P(A)$ determine $tp^{\mathcal{L}_P}(A)$.

$\therefore$ a free extension of lovely pairs is elementary.

Continuing proof from last time:

④ $\Rightarrow$ ①: Let $(M,P) \subseteq (N,P)$ be free ie $M \prec N$ and $M \underset{P(M)}{\downarrow} P(N)$

It suffices to prove that $\forall (M',P) \preceq (M,P)$

s.t. $|M'| \leq |T|$: $(M',P) \preceq (N,P)$ (by Lowenheim-Skolem).

So we may assume $|(M,P)| \leq |T|$

$[\ (M',P) \leq (M,P) \Rightarrow M' \underset{P(M')}{\downarrow} P(M) \Rightarrow M' \underset{P(M')}{\downarrow} P(N)\ ]$.

Since $\exists$ a lovely pair that is $|T|^+$-saturated, call it (K,P),

and since T_P is complete: we may assume $(M,P) \underset{\text{elementary}}{\hookrightarrow} (K,P)$.

Finally we may assume (N,P) is $|K|^+$-lovely.

1. $(M,P) \preceq (K,P) \Rightarrow M \underset{P(M)}{\downarrow} P(K)$

Since (N,P) is $|K|^+$-lovely, we may realise

$tp(P(K)/M)$ inside $P(N)$ (coheir prop.)

Call the realisation $P(K)$.

Now we may realise $tp(K/M \cup P(K))$ in N st.

$K \underset{M \cup P(K)}{\downarrow} P(N)$ ⊛ (extn prop.)

So $P(M) \subseteq P(K) \subseteq P(N)$.

$$M \underset{P(M)}{\downarrow} P(N) \Rightarrow M \underset{P(K)}{\downarrow} P(N)$$

$$\Rightarrow K \underset{P(K)}{\downarrow} P(N) \quad \text{by } \circledast.$$

But K, N are lovely : $(M,P) \preceq (K,P) \preceq (N,P)$. $\square$

Viewing this theorem:

Good: The saturated models of T_P are precisely the lovely pairs.

Being a lovely pair is "first order"

in terms that they are precisely the sat. models.

only this works

Bad: The class of lovely pairs is not "first order."

<u>Theorem</u> : There always exists a cat T_P whose saturated models are the lovely pairs.

In fact, we don't need to assume that T is f.o. : this works for every simple thick T.

Good: T_P is f.o.

Bad: T_P is a non-f.o. cat. (not too bad)

Pf. Assume $a \in (M,P) \models T_P$. [if you want, assume f.o.]

Define $a^c := Cb(a/P(M))$ Does not depend on M:
M is free in $N \Rightarrow a \underset{P(M)}{\downarrow} P(N)$

so canonical bases are the same.

Claim: $a^c \in dcl^P(a)$: an automorphism fixing a pointwise & $\underset{P(M)}{\ }$ setwise fixes $Cb(a/\underset{P(M)}{P})$

So $tp^P(a)$ determines $tp^P(\underbrace{a, a^c}_{\hat{a}})$ and therefore $tp(\underbrace{a, a^c}_{\hat{a}})$

Note: here $dcl^P, tp^P, \ldots$ mean in the sense of T_P.

On the other hand, $\hat{a}$ is free: $a \underset{a^c}{\downarrow} P \Rightarrow \hat{a} \underset{a^c}{\downarrow} P$.

So $tp(\hat{a})$ determines $tp^P(\hat{a})$ and therefore $tp^P(a)$.

(cheating since may contain hyperimaginary, but still works)

$\therefore \quad tp^P(a) \overset{\sim}{\longleftrightarrow} tp(\hat{a})$

If T_P is f.o. : Assume $\varphi(x,y) \in \mathcal{L}$ and $a \in (M,P) \models T_P$.

Then: $M \models \exists y \in P$ s.t. $\varphi(a,y) \Leftrightarrow \varphi(a,y)$ dnd $/P(M)$
(⇐ by cohein)

Exercise
$\Longleftrightarrow \varphi(a,y)$ dnd $/ a^c$

<u>Fact</u> T_P admits QE up to boolean combinations of "$\exists y \in P\ \varphi(x,y)$".

<u>Sketch of Proof</u> Assume a, b both satisfy same formulas of this kind.

Let $tp_\Delta(a) := \{ \exists y \in P\ \varphi(x,y) : \exists y \in P\ \varphi(a,y) \}$
$\cup \{ \neg \exists y \in P\ \varphi(x,y) : \neg \exists y \in P\ \varphi(a,y) \}$

Assume that $a \in (M,P)$, $b \in (N,P)$, $tp_\Delta(a) = tp_\Delta(b) =: q(x)$.

Then $q(x) \cup \{$ a copy of $P(M) \subseteq P \} \cup$
$\{$ a copy of $P(N) \subseteq P \}$ is consistent

$x \models tp(a/P(M))$ $\quad x \models tp(b/P(N))$

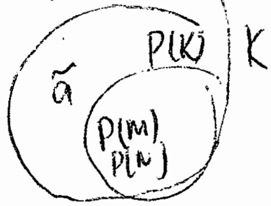
an $\mathcal{L}_P$-type with constants for $P(M)$, $P(N)$.

$c_t : t \in P(M)$ $\quad d_s : s \in P(N)$.

$q(x) \wedge \bigwedge P(c_t) \wedge \bigwedge P(d_s) \wedge x, \bar{c} \models tp(a, P(M)) \wedge$
$x, \bar{d} \models tp(b, P(N))$

finitely realisable: $\varphi(x, c)$ in and $\psi(x, d)$ in
$\Rightarrow q \vdash \exists y, z \in P\ \varphi(x,y) \wedge \psi(x,z)$

$\therefore$ consistent.



Since $\tilde{a}$ satisfies the negative part of q:

$tp(\tilde{a}/P(K))$ is a coheir of $tp(\tilde{a}/P(M))$, $tp(\tilde{a}/P(N))$

$\tilde{a} \underset{P(M)}{\downarrow} P(K)$ & $\tilde{a} \underset{P(N)}{\downarrow} P(K)$

$Cb(\tilde{a}/P(M)) = Cb(\tilde{a}/P(K)) = Cb(\tilde{a}/P(N)) = \tilde{a}^c$

so $a, a^c \equiv \tilde{a}, \tilde{a}^c \equiv b, b^c$

so $a \equiv^P b \quad \rightsquigarrow$ QE.

<u>Theorem</u> Let $a, b, c \in (M, P) \models T_P$.

Let $(a_i, b_i, c_i : i < \omega)$ be a Morley seq for $a, b, c / P(M)$ (in sense of T).

Then TFAE: (1) $a \underset{cP}{\downarrow} b$ and $(ac)^c \underset{c^c}{\downarrow} (bc)^c$

doesn't matter what M is again.

(2) $a \underset{cP}{\downarrow} b$ and $\widehat{ac} \underset{\hat{c}}{\downarrow} \widehat{bc}$

(3) $(a_i : i<\omega) \underset{(c_i : i<\omega)}{\downarrow} (b_i : i<\omega)$

Call these notions: $a \underset{c}{\downarrow}^P b$

So it follows immeditely from ③ that $\mathop{\perp\!\!\!\perp}^P$ satisfies all axioms for independence except maybe indep. thm.

<u>Prop</u>: Assume $a_1 \mathop{\perp\!\!\!\perp}^P_c a_2$ and $b_i \mathop{\perp\!\!\!\perp}^P_c a_i$ $\forall i \in \{1,2\}$ and $b_1 \equiv^P_{bdd^T(\hat{c})} b_2$. Then $\exists b$ s.t. $b \mathop{\perp\!\!\!\perp}^P_c a_1 a_2$ s.t. $b \equiv^P_{a_i\, bdd(\hat{c})} b_i$

$\therefore$ T_P is simple and $\mathop{\perp\!\!\!\perp}^P$ = nondividing and
$bdd^P(c) = dcl^P(bdd^T(\hat{c}))$.

(Hand waving) We said $a \equiv^P b \iff \exists$ aut. sending $\hat{a}$ to $\hat{b}$
$\iff \exists$ aut. sending parallelism class of $tp(a/p)$ to that of $tp(b/p)$.

T_P-types are the same thing as types of T-parallelism classes.

<u>Example</u>: $U(ACF_P) = \omega$ $U(\text{vector space } p) = 2$.
But $U(ACF) = U(\text{vector space}) = 1$.