Lecture 19

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1 Special cases of submodular flows

We saw last time that orientation of a $2k$-edge connected graph into a k-arc connected digraph and the Lucchesi and Younger Theorem were special cases of submodular flows. Other familiar problems can also be phrased as submodular flows.

1.1 Example: Circulation

Let $\mathcal{C} = 2^V \setminus \{\emptyset, V\}$ and let f be identically zero. Then for any $U \in \mathcal{C}$,

$$\begin{aligned} x(\delta^{in}(U)) - x(\delta^{out}(U) &\leq 0 \\ x(\delta^{in}(V \setminus U)) - x(\delta^{out}(V \setminus U) &\leq 0 \end{aligned} \tag{1}$$

which implies that $x(\delta^{in}(U)) = x(\delta^{out}(U)$ for all $U \subset V$. In particular, we have that $x(\delta^{in}(v)) = x(\delta^{out}(v))$ for all $v \in V$. In this case, the submodular flow reduces to the circulation problem from network flows.

1.2 Example: Matroid Intersection

Given two matroids on the same ground set, $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, let S_1 and S_2 be identical copies of S and let V=$S_1 \cup S_2$.

Consider the collection

$$\mathcal{C} = \{U \subset V, U \neq \emptyset : U \subseteq S_1 \text{ or } S_1 \subseteq U\}\,. \tag{2}$$

If A and B are elements of $\mathcal{C}$ with nonempty intersection and $A \cup B \neq V$, then

- $S_1 \subseteq A$, $S_1 \subseteq B \implies S_1 \subseteq A \cap B$ and $S_1 \subseteq A \cup B$
- $S_1 \subseteq A$, $B \subseteq S_1 \implies A \cap B = B \subseteq S_1$ and $S_1 \subseteq A \cup B = A$
- $A \subseteq S_1$, $B \subseteq S_1 \implies A \cap B \subseteq S_1$ and $A \cup B \subseteq S_1$

and hence for all cases, $A \cup B$ and $A \cap B$ are in $\mathcal{C}$ proving that $\mathcal{C}$ is a crossing family.

Let

$$f(U) = \begin{cases} r_1(U) & U \subseteq S_1 \\ r_2(V \setminus U) & S_1 \subseteq U\,. \end{cases} \tag{3}$$

f is readily seen to be crossing submodular by checking cases. If $A \subseteq S_1$, $B \subseteq S_1$ then the submodular inequality for f follows from submodularity of r_1. If $S_1 \subseteq A$, $S_1 \subseteq B$ then by deMorgan's laws

$$\begin{aligned} V \setminus (A \cup B) &= (V \setminus A) \cap (V \setminus B) \\ V \setminus (A \cap B) &= (V \setminus A) \cup (V \setminus B)\,. \end{aligned} \tag{4}$$

Therefore, submodularity of f follows from the submodularity of r_2. Finally, if $S_1 \subseteq A$, $B \subseteq S_1$ then $f(A \cap B) = r_1(A \cap B)$ and $f(A \cup B) = r_2(V \setminus (A \cup B))$. Since $|B| \geq |A \cap B|$ and $|V \setminus A| \geq |V \setminus (A \cup B)|$, the submodular inequality holds here as well.

To define the arc set on V, let arcs connect the elements in S_2 to their bijective copy in S_1. That is,

$$A = \{(s_2, s_1) : s \in S\} \tag{5}$$

Now consider the submodular flow constraints. On this graph $\delta^{out}(U) = \emptyset$ for all $U \in \mathcal{C}$. If $U \subset S_1$,

$$x(U) = x(\delta^{in}(U)) \leq r_1(U)\,. \tag{6}$$

If $S_1 \subset U$, $U = V \setminus U'$ and hence

$$x(U') = x(\delta^{in}(U)) \leq r_2(U')\,. \tag{7}$$

Putting this all together, we find that the submodular flow polytope is defined by

$$\left\{ \begin{array}{ll} x(U) \leq r_1(U) & \forall\, U \subset S \\ x(U) \leq r_2(U) & \forall\, U \subset S \\ x_s \geq 0 & \forall\, s \in S \end{array} \right\} \tag{8}$$

which is the matroid intersection polytope.

We will see shortly that the proof of the Edmonds and Giles theorem will use similar techniques as those presented in the proof of the total dual integrality of the matroid intersection polytope.

2 Proof of Edmonds and Giles Theorem

We will prove the Edmonds and Giles theorem by showing that the optimal solution is defined by a totally unimodular system of equations. Here the particular notion we will exploit is

Definition 1 *A set $\mathcal{F} \subseteq 2^V$ is* cross-free *if for any F_1, F_2 in $\mathcal{F}$, either $F_1 \subseteq F_2$, $F_2 \subseteq F_1$, $F_1 \cap F_2 = \emptyset$ or $F_1 \cup F_2 = V$.*

We can now proceed to prove the following

Theorem 1 (Edmonds-Giles) *Let $\mathcal{C}$ be a crossing family on V, let $f : \mathcal{C} \to \mathbb{R}$ be crossing submodular, then the polytope*

$$\left\{ \begin{array}{ll} x(\delta^{in}(U)) - x(\delta^{out}(U)) \leq f(U) & \forall\, U \in \mathcal{C} \\ d_a \leq x_a \leq c_a & \forall\, a \in A \end{array} \right. \tag{9}$$

is totally dual integral.

Proof: Take w to be an integral vector and consider the linear program

$$\begin{array}{lll} \max & \sum_a w_a x_a & \\ \text{s.t.} & x(\delta^{in}(U)) - x(\delta^{out}(U)) \leq f(U) & \forall\, U \in \mathcal{C} \\ & d_a \leq x_a \leq c_a & \forall\, a \in A. \end{array} \tag{10}$$

The associated dual problem is

$$\begin{array}{lll} \min & \sum_{U \in \mathcal{C}} f(U) y_U - \sum_a d_a s_a + \sum_a c_a t_a & \\ \text{s.t} & \sum_{\substack{U \\ u \notin U \\ v \in U}} y_U - \sum_{\substack{U \\ u \in U \\ v \notin U}} y_U - s_a + t_a = w_a & \forall\, a = (u, v) \in A \\ & y_U \geq 0,\ s_a \geq 0\, t_a \geq 0. & \end{array} \tag{11}$$

We will now construct an optimum dual solution such that the set $\mathcal{F} = \{U : y_U > 0\}$ is cross-free. Suppose there are sets F_1 and F_2 in $\mathcal{F}$ such that $F_1 \not\subseteq F_2$, $F_2 \not\subseteq F_1$, $F_1 \cap F_2 \neq \emptyset$ and $F_1 \cup F_2 \neq V$.

Let $\epsilon = \min\{y_A, y_B\}$ and define a new dual vector y' as

$$y'_T = \begin{cases} y_T - \epsilon & T = F_1 \text{ or } T = F_2 \\ y_T + \epsilon & T = F_1 \cap F_2 \text{ or } T = F_1 \cup F_2 \\ y_T & \text{otherwise} \end{cases} \tag{12}$$

y' is readily seen to be feasible as any decrease in the value of y_{F_1} or y_{F_2} is matched by an increase in the value of $y_{F_1 \cup F_2}$ and $y_{F_1 \cap F_2}$. Furthermore, y' is optimal as

$$c(y') = c(y) - \epsilon[f(F_1) + f(F_2) - f(F_1 \cap F_2) - f(F_1 \cup F_2)] \leq c(y) \tag{13}$$

We can repeat this process of eliminating crosses until we are left with a cross free family. The process terminates because if we consider the potential function

$$\psi(y) = \sum_{U \in \mathcal{C}} y_U |U||V \setminus U| \tag{14}$$

then $\psi(y') \leq \psi(y)$.

It suffices to show that the matrix defined by

$$\sum_{U: u \notin U, v \in U} y_U - \sum_{U: u \in U, v \notin U} y_U \qquad U \in \mathcal{F} \tag{15}$$

is totally unimodular when $\mathcal{F}$ is cross-free. This follows from

Theorem 2 *Let $\mathcal{F}$ be a cross-free family on 2^V. Let M be an $|A| \times |\mathcal{F}|$ matrix such that column f is the vector $\chi^{\delta^{in}(U)} - \chi^{\delta^{out}(U)}$. Then M is totally unimodular.*

which is proved in chapter 13 of Shrijver and follows from an inductive argument similar to the one presented for matroid intersection.

□

3 Algorithms for submodular flows

Knowing that the submodular flow polytope is totally dual integral does not explicitly tell us how to optimize over it. However, optimization can be performed in polynomial time using submodular function minimization. Given an $x \in \mathbb{R}^{|V|}$ the function

$$f(U) - x(\delta^{in}(U)) + x(\delta^{out}(U)) \tag{16}$$

is submodular on $\mathcal{C}$. This is because

$$g(U) = -x(\delta^{in}(U)) + x(\delta^{out}(U)) \tag{17}$$

is *modular*. Indeed, for $A, B \subset V$

$$x(\delta^{in}(A)) + x(\delta^{in}(B)) = x(\delta^{in}(A \cap B)) + x(\delta^{in}(A \cup B)). \tag{18}$$

and similar equality holds for δ^{out}. Since minimizing a submodular function can be performed in polynomial time, we can compute

$$\min_{U \in \mathcal{C}} f(U) - x(\delta^{in}(U)) + x(\delta^{out}(U)) \tag{19}$$

efficiently. This provides the minimal violated constraint which we can use to construct a separating hyperplane and run the the ellipsoid algorithm.

This method is not efficient in practice, so typically submodular flows are solved via a reduction to polymatroid intersection. Polymatroids are generalizations of matroids where the rank of a set may be larger than the cardinality of that set. But there are special cases where even this is machinery not necessary. One such example is orienting a $2k$-edge connected graph into a k-arc connected digraph where we only need to employ matroid intersection.

3.1 Orienting a $2k$-edge connected graph

Let S denote the set of all pairs of vertices (u, v). Construct two matroids with ground set S as follows. Let M_1 be the partition matroid which allows only one of (u, v) or (v, u) to be selected. Define the bases for M_2 to be sets $B \subset S$ such that $|B| = |E|$ and for U a nonempty subset of V which does not equal V

$$|B \cap H(U)| \leq |E(U)| + |\delta(U)| - k \tag{20}$$

here $H(U) = \{(u, v) \in S : v \in U\}$.

We assert the following to be proved next time

Proposition 3 *M_2 is a matroid*

Proposition 4 *Testing independence in M_2 can be performed by network flows.*

The theorem of Nash-Williams from last time shows that these two matroids have a common basis (and the minmax relation for matroid intersection would also show it). Assuming the two propositions, it follows immediately that we can find this basis using matroid intersection. It is similarly immediate that a common basis for M_1 and M_2 is an orientation of G which is k-arc connected.