PROBLEMS ON SUMS AND INTEGRALS

Contents

1 Swapping sums

1.1 Finite sums

Here is an example that many of you might already know from high school math contests.

Example 1. Let n be a positive integer. Prove that

$$\sum_{k\ge 1} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{1}{2}n(n+1).$$

Proof. The key idea is to rewrite the floor as a sum involving divisors:

$$\sum_{k\ge 1} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor = \sum_{k\ge 1} \varphi(k) \sum_{\substack{k\mid m \\ k\le n}} 1 = \sum_{k\ge 1} \sum_{\substack{k\mid m \\ m\le n}} \varphi(k).$$

Thus we're computing the sum of $\varphi(k)$ over several pairs of integers (k,m) for which $k \mid m$, $m \le n$. For example, if $n = 6$, the possible pairs (k,m) are given be the following table:

$$(k,m) \in \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ & (2,2) & & (2,4) & & (2,6) \\ & & (3,3) & & & (3,6) \\ & & & (4,4) & & \\ & & & & (5,5) & \\ & & & & & (6,6) \end{array} \right\}$$

Nominally, we're supposed to be summing by the rows of this table (i.e. fix k and run the sum over corresponding m). However, by interchanging the order of summation we can instead consider this as a sum over the rows: if we instead pick the value of m first, we see that

$$\sum_{k\geq 1}\sum_{\substack{k|m \\ m\leq n}}\varphi(k)=\sum_{m=1}^{n}\sum_{k|m}\varphi(k).$$

Using the famous fact $\sum_{d|n}\varphi(d)=n$, we conclude

$$\sum_{m=1}^{n}\sum_{k|m}\varphi(k)=\sum_{m=1}^{n}m=\frac{1}{2}n(n+1).$$

$\square$

Here one has the idea that one can "swap the order of summation": even though there is a single $\sum$ initially, by rewriting it as a double $\sum$ and then swapping the order, we are able to solve the problem.

The goal of this lecture is to try and push this idea to allow us to do similar calculations over both infinite sums and integrals. Because of the introduction of infinity, things become a little more complicated and some more care is necessary. So, in the first part of the lecture we will address conditions on which rearranging the order of summation or integration is permissible. After that we will see several applications.

1.2 Absolute and conditional convergence

Let $\sum_n a_n$ be an infinite series of complex numbers; then its limit is defined as

$$\sum_n a_n := \lim_{N\to\infty}\left(\sum_{n=1}^{N}a_n\right).$$

Note that this depends on the order of the terms: if we permute the sequence, the limit might change! This is weird and bad since we would want "infinite addition" to be commutative, so we want a way to avoid this behavior. This is accomplished by using the so-called notion of absolute convergence.

Definition 2. If $\sum a_k$ converges, we say it *converges absolutely* if $\sum |a_k| < \infty$, and *converges conditionally* otherwise.

Theorem 3 (Rearrangement okay iff absolutely convergent). *Let $\sum a_n$ be a convergent series of complex numbers.*

(a) If $\sum a_n$ is absolutely convergent, it is invariant under permutation of the terms (the sum will still converge, and the limit remains the same).

(b) If $\sum a_n$ is conditionally convergent and a_n are real numbers, then there exists a permutation of the terms for which the sum converges to 2018.

Thus, any time before you try to rearrange the series, you must check first that it's absolutely convergent. With two $\sum$ signs the statement reads:

Theorem 4 (Fubini for doubly-indexed infinite sums)**.** *Let $a_{m,n} \in \mathbb{C}$. If any of the three quantities*

$$\sum_{(m,n)\in\mathbb{N}^2} |a_{m,n}|, \qquad \sum_m \left(\sum_n |a_{m,n}| \right), \qquad \sum_n \left(\sum_m |a_{m,n}| \right)$$

are convergent, then

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} = \sum_m \left(\sum_n a_{m,n} \right) = \sum_n \left(\sum_m a_{m,n} \right)$$

and all three series are convergent.

Corollary 5 (Tonelli for doubly-indexed infinite sums)**.** *Let $a_{m,n} \in \mathbb{R}_{\ge 0}$. Then*

$$\sum_{(m,n)\in\mathbb{N}^2} a_{m,n} = \sum_m \left(\sum_n a_{m,n} \right) = \sum_n \left(\sum_m a_{m,n} \right)$$

where we allow the possibility all three diverge.

Here is the classic example.

Example 6 (Putnam 2016 B6). Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1}.$$

Proof. Before anything else, the sum is absolutely convergent since we have

$$\sum_{\substack{k\ge 1\\ n\ge 0}} \frac{1}{k(k\cdot 2^n+1)} < \sum_{k\ge 1} k^{-2} \sum_{n\ge 0} \frac{1}{2^n} = \frac{\pi^2}{6}\cdot 2 < \infty.$$

Thus we may swap the order of summation freely.

We use $d = k\cdot 2^n + 1 \ge 2$ as the summation variable, so that the sum in question is

$$\sum_{d\ge 2} \frac{1}{d} \sum_{\substack{k\\ \exists n: d-1=k\cdot 2^n}} \frac{(-1)^{k-1}}{k}.$$

Now we claim that the inner sum is exactly $\frac{1}{d-1}$. Indeed, if $d-1 = 2^r m$ with m odd, then the sum is

$$\begin{aligned} \frac{(-1)^{m-1}}{m} + \frac{(-1)^{2m-1}}{2m} + \dots + \frac{(-1)^{2^r m-1}}{2^r m} &= \frac{1}{m}\left(\frac{1}{1} - \frac{1}{2} - \dots - \frac{1}{2^r}\right) \\ &= \frac{1}{2^r m} \\ &= \frac{1}{d-1}. \end{aligned}$$

Consequently, the final answer is

$$\sum_{d\ge 2} \frac{1}{d(d-1)} = \sum_{d\ge 2}\left(\frac{1}{d-1} - \frac{1}{d}\right) = 1.$$

$\square$

2 Riemann integral

So far all of this is fair-game on high school. We'll now move into the realm of calculus.

Definition 7. A *tagged partition* P of $[a,b]$ consists of a partition of $[a,b]$ into n intervals, with a point ξ_i in the nth interval, denoted

$$a \le x_0 < x_1 < x_2 < \cdots < x_n \le b \qquad \text{and} \qquad \xi_i \in [x_{i-1}, x_i] \quad \forall\, 1 \le i \le n.$$

The *mesh* of P is the width of the longest interval, i.e. $\max_i (x_i - x_{i-1})$.

Theorem 8 (Riemann integral)**.** *Let $f \colon [a,b] \to \mathbb{C}$ be continuous. Then the definition*

$$\int_a^b f(x)\, dx = \lim_{\substack{P \text{ tagged partition} \\ \text{mesh } P \to 0}} \left(\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right)$$

is well-defined (and finite).

There are a bunch of remarks I want to make about this result.

2.1 Compactness and improper integrals

We won't prove the definition of the Riemann integral works out, but we will mention that its proof hinges crucially on:

Fact 9. The interval $[a,b]$ is *compact*, so continuous functions $f \colon [a,b] \to \mathbb{C}$ behave well. In particular, f is bounded, and "uniformly continuous".

This fact is false for open (or unbounded) intervals: consider the function $1/\sqrt{x}$ on $(0,1)$, for example. This gives rise to the notion of "improper integrals", such as

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx.$$

As written, this does not officially make sense as a Riemann integral, since $f(x) = \frac{1}{\sqrt{x}}$ is not a function on $[0,1]$. Rather, we implicitly mean

$$\lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\, dx$$

since $f(x)$ *is* well-defined on $[\varepsilon, 1]$. In this case, there is no guarantee the limit exists; for example $\int_0^1 x^{-1}\, dx = \infty$.

Similarly, it's possible to set endpoints at ∞ by e.g.

$$\int_{-\infty}^{\infty} f(x) \coloneqq \lim_{B \to \infty} \int_{-B}^{B} f(x)$$

for example.

2.2 Mesh sums

Sometimes, you will find that a sum can be written in such a way that it corresponds to the mesh of a Riemann integral. In that case, one is very happy, because then it turns the entire sum into a single integral!

Example 10. Evaluate

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right).$$

Proof. Write as

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\right).$$

Then, this is a mesh sum for $f(x)=\frac{1}{1+x}$ over $[0,1]$. Thus by definition it approaches $\int_0^1 \frac{1}{1+x}\,dx = \log 2$. $\square$

2.3 Discretization and inequalities

If asked to prove an identity or inequality about integrals, it is often possible to revert back to a discrete sum, a technique called *discretization*. For example, suppose one wishes to prove the Cauchy-Schwarz inequality in the form

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right)$$

for continuous functions $f,g\colon [a,b]\to\mathbb{R}$. By taking meshes, it is sufficient to prove

$$\left(\frac{1}{n}\sum_i f(a_i)g(a_i)\,dx\right)^2 \le \left(\frac{1}{n}\sum_i f(a_i)^2\,dx\right)\left(\frac{1}{n}\sum_i g(a_i)^2\,dx\right).$$

which is of course just the classical Cauchy-Schwarz from high school.

In practice, aside from discretization, most integral inequalities on competitions will really just use the following fact:

Lemma 11 (The obvious inequality)**.** *Let $f,g\colon [a,b]\to\mathbb{R}$ be continuous functions. If $f(x)\le g(x)$ for all $x\in[a,b]$ then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

3 Lebesgue integrals

3.1 Advantages of Lebesgue integrals

Unfortunately, Riemann integrals are terrible. In order to properly state theorems about interchanging order of summation, it'll be much more convenient to proceed with the **Lebesgue integral**, which I will generally denote by $\int_X$ to distinguish it from the Riemann integral $\int_a^b$.

Defining the Lebesgue integral is much more involved, because it involves a bunch of measure theory, so I *won't* define what it is (but those of you taking 18.175 will find out really soon). However, I'll at least mention the following reasons it's appreciated.

- **Better theorems about swapping limits and sums**. For example, for the Riemann integral, swapping $\sum_n \int_a^b f_n$ and $\int_a^b \sum_n f_n$ requires *uniform convergence*, which is a pretty strong condition (although it'll be true for Taylor series, which is a frequent use case for us).

- **Improper integrals can be handled natively**. You can write $\int_{(0,1)} \frac{1}{\sqrt{x}}\, dx$ and $\int_{\mathbb{R}} \exp(-x^2)\, dx$ and it makes sense, unlike for the Riemann case where one has to use an improper integral.

- **More versatile**. Although we won't encounter any, some functions that were previously not Riemann integrable can now be assigned values. The classic example is $\int_{[0,1]} \mathbf{1}_{\mathbb{Q}} = 0$.

3.2 Riemann integrals and Lebesgue integrals

Of course, it'd be really silly if there wasn't some guarantee that the Riemann integrals and Lebesgue integrals agree.

The rules for converting a Riemann and Lebesgue integral are as follows:

- For continuous functions $f\colon [a,b] \to \mathbb{C}$, the Riemann integral and Lebesgue integrals coincide. So **proper Riemann integrals work out of the box**.

- For continuous nonnegative functions $f\colon (a,b) \to \mathbb{R}_{\ge 0}$ on an open (or half-open) interval where one needs improper integrals, the improper Riemann integral and Lebesgue integrals coincide (where we allow the possibility that the integrals are both $+\infty$). Here, $a = -\infty$ and $b = +\infty$ are allowed too.

- For general $f\colon (a,b) \to \mathbb{C}$, if the partial integrals $\int_c^d |f|\, dx$ are bounded for any $[c,d] \subset (a,b)$ then we can also swap as above.

On the other hand, if your signs are all over the place, then there isn't hope in general of converting improper Riemann integrals to Lebesgue ones. A famous textbook example is $\int_0^\infty \frac{\sin x}{x}\, dx$ which in fact is *not* covered by Lebesgue integration.

3.3 Swapping double integrals

I'll state this in the full generality, though we'll only use it in the cases where the "σ-finite measure spaces" are $\mathbb{N}$ (corresponding to infinite sums) or sub-intervals of $\mathbb{R}$ (corresponding to Riemann integrals).

Theorem 12 (Fubini)**.** *Let X and Y be "σ-finite measure spaces". Let $f\colon X \times Y \to \mathbb{C}$ be continuous (or just "measurable"). If any of $\int_X \left(\int_Y |f(x,y)|\, dy\right)\, dx$, $\int_Y \left(\int_X |f(x,y)|\, dx\right)\, dy$, $\int_{X\times Y} |f(x,y)|\, d(x,y)$ are finite, then we have*

$$\int_X \left(\int_Y f(x,y)\, dy\right)\, dx = \int_Y \left(\int_X f(x,y)\, dx\right)\, dy = \int_{X\times Y} f(x,y)\, d(x,y).$$

Corollary 13 (Tonelli)**.** *Let X and Y be "σ-finite measure spaces". Let $f\colon X \times Y \to \mathbb{R}_{\ge 0}$ be continuous (or just "measurable") and nonnegative. Then*

$$\int_X \left(\int_Y f(x,y)\, dy\right)\, dx = \int_Y \left(\int_X f(x,y)\, dx\right)\, dy = \int_{X\times Y} f(x,y)\, d(x,y)$$

where we allow the possibility that all three are $+\infty$.

Remark 14.

- If $X = \mathbb{N}$ and $Y = \mathbb{N}$, then this corresponds to the double sums we stated earlier.
- If $X = \mathbb{N}$ and $Y \subset \mathbb{R}$ is an interval, then this states that $\sum \int$ and $\int \sum$ can be swapped.
- Note that if X and Y are finite closed intervals and $f\colon X \times Y \to \mathbb{C}$ is continuous, then hypotheses of Fubini are automatically satisfied, since $X \times Y$ is compact. The situation where X and Y are open/infinite is more slippery, although in most cases we'll have nonnegativity and then Tonelli will save us.

Tonelli's theorem (together with the result that even improper Riemann integrals are okay with nonnegative functions) means that whenever you have nonnegative functions, you can proceed *no holds barred* — everything works beautifully. In other words **nonnegative $\implies$ euphoria**.

3.4 Interchanging limits and Lebesgue integrals

You can read this off of the results on sums, but we'll state them here since they have names.

Theorem 15 (Dominated convergence theorem)**.** *Let $f_n\colon I \to \mathbb{C}$ be a sequence of continuous functions on an interval $I \subseteq \mathbb{R}$. Assume that $|f_n(x)| \le g(x)$ for all x, where $\int_I g(x) < \infty$ (i.e. g is integrable). Then $\lim_n f_n(x)$ is integrable and*

$$\lim_{n\to\infty} \int_I f_n(x)\, dx = \int_I \lim_{n\to\infty} f_n(x)\, dx.$$

Theorem 16 (Monotone convergence theorem)**.** *Suppose that $f_n\colon I \to \mathbb{R}_{\ge 0}$ is a sequence of continuous functions on an interval $I \subseteq \mathbb{R}$ which are also nonnegative. Assume further that $f_n(x) \le f_{n+1}(x)$ for $n \in \mathbb{N}$, $x \in I$. Then*

$$\lim_{n\to\infty} \int_I f_n(x)\, dx = \int_I \lim_{n\to\infty} f_n(x)\, dx$$

where the value of any of these integrals is allowed to be infinite.

4 Techniques for introducing more sums

4.1 Taylor series

Some common ones:

$$\begin{aligned}
\exp(x) &= \sum_{n\ge 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots && \forall\, x \in \mathbb{R} \\
\log(1-x) &= -\sum_{n\ge 1} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots && \forall\, |x| < 1 \\
\frac{1}{1-x} &= \sum_{n\ge 0} x^n = 1 + x + x^2 + \dots && \forall\, |x| < 1 \\
\arctan(x) &= \sum_{n\ge 0} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots && \forall\, |x| < 1.
\end{aligned}$$

There is a nice theorem about Taylor series in general:

Theorem 17 (Convergence of Taylor series)**.** *Let f be an analytic function. Within its radius of convergence, the Taylor series for f will*

- converge absolutely *for any x as a series of complex numbers, and*
- converge uniformly *on any compact sub-interval, as a series of functions (i.e. it is compactly convergent).*

I mention uniform convergence here since it's actually strong enough to allow swapping integration even for the Riemann integral. Here's the definition:

Definition 18. A sequence of functions $F_n \colon [a, b] \to \mathbb{C}$ is said to *converge uniformly* to the function $F \colon [a, b] \to \mathbb{C}$ if

$$\lim_{n\to\infty} \sup_{x\in[a,b]} |F_n(x) - F(x)| = 0.$$

A series $\sum_n f_n$ converges uniformly if its partial sums $F_n = \sum_{k=1}^n f_k$ do.

But we'll be mostly using Lebesgue integrals anyways.

So, whenever you have an analytic function on a closed interval, all the summation results work fine! Here is a very famous example.

Example 19. Compute

$$\int_0^1 \log x \log(1-x) \, dx.$$

There is some subtlety here since this integral looks like it might be improper! Fortunately, it's not quite, since $\lim_{x\to 0^+} \log(x)\log(1-x) = 0$, and in this way we can actually regarding $\log(x)\log(1-x)$ as a proper integral on $[0, 1]$.

Proof. Switch to Lebesgue integration. The integral is then

$$\begin{aligned}
I &= -\int_{[0,1]} \log x \sum_{n\ge 1} \frac{x^n}{n} \, dx \\
&= -\sum_{n\ge 1} \frac{1}{n} \int_0^1 x^n \log x \, dx && \text{(by Tonelli)} \\
&= -\sum_{n\ge 1} \frac{1}{n} \left[x^{n+1} \cdot \frac{(n+1)\log x - 1}{(n+1)^2} \right]_{x=0}^{x=1} \text{(integration by parts)} \\
&= -\sum_{n\ge 1} \frac{1}{n} \left[x^{n+1} \cdot \frac{(n+1)\log x - 1}{(n+1)^2} \right]_{x=0}^{x=1} \\
&= \sum_{n\ge 1} \frac{1}{n(n+1)^2} \\
&= \sum_{n\ge 1} \left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right].
\end{aligned}$$

The Nth partial sum of this is equal to $1 - \frac{1}{N+1} - \sum_{n=1}^{N} \frac{1}{(n+1)^2}$ which gives $2 - \frac{\pi^2}{6}$ as $N \to \infty$. $\square$

Remark 20 (An application of Feynman's trick). In my original notes, I had obtained the identity $\int_0^1 x^n \log x \, dx = -\frac{1}{(n+1)^2}$ using integration by parts. In class it was pointed out that *Feynman's* *trick*, more descriptively called "differentiating under the integral sign", gives a shorter way to prove this. Start by writing

$$\int_0^1 x^n \, dx = \frac{1}{n+1}$$

and then treat $n \in \mathbb{R}$ as a parameter. This allows one to differentiate both sides with respect to n, yielding

$$\begin{aligned} \int_0^1 \frac{d}{dn} x^n \, dx &= \frac{d}{dn} \frac{1}{n+1} \\ \implies \int_0^1 x^n \log x \, dx &= -\frac{1}{(n+1)^2}. \end{aligned}$$

See `http://www.math.uconn.edu/~kconrad/blurbs/analysis/diffunderint.pdf` for more details on this trick.

4.2 Eliminating fractions

The following seemingly obvious statement is surprisingly useful.

Lemma 21 (Denominator $\to$ integral)**.** *For any real number $s > -1$ we have*

$$\frac{1}{s+1} = \int_{(0,1)} t^s \, ds.$$

As a simple use case, let's suppose we were given $\sum_n \frac{x^n}{n}$ for some $|x| < 1$ and wanted to figure out what function it was (without knowing anything about log in advance). We can write

$$\begin{aligned} \sum_n \frac{x^n}{n} &= \sum_n x^n \int_{[0,1]} t^{n-1} \, dt \\ &= \sum_n \int_{[0,1]} x(xt)^{n-1} \, dt \\ &= \int_{[0,1]} \sum_n x(xt)^{n-1} \, dt \\ &= \int_{[0,1]} \frac{x}{1-xt} \, dt \\ &= [-\log(1-xt)]_{t=0}^1 \\ &= [-\log(1-xt)]_{t=0}^1 = -\log(1-x). \end{aligned}$$

Let's also see a solution to the earlier double sum.

Example 22 (Putnam 2016 B6)*.* Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n + 1}.$$

Proof. Check conditional convergence of the double sum in the same way as before. Thus we apply Fubini freely:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \frac{1}{k2^n+1} &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=0}^{\infty} \int_{[0,1]} t^{k2^n}\, dt \\
&= \int_{[0,1]} \left(- \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(-t^{2^n})^k}{k} \right) dt \\
&= \int_{[0,1]} \sum_{n=0}^{\infty} \log(1+t^{2^n})\, dt = \int_{[0,1]} \log\left(\prod_{n=0}^{\infty} (1+t^{2^n}) \right) dt \\
&= \int_{[0,1]} \log\left(\frac{1}{1-t} \right) dt = \int_{[0,1]} -\log(1-t)\, dt = 1.
\end{aligned}$$

□

4.3 Fourier series

If $f\colon \mathbb{R} \to \mathbb{C}$ is continuous with period 1, then

$$f(x) = \lim_{N\to\infty} \sum_{m=-N}^{N} a_m \exp(2\pi i m x).$$

The Fourier coefficients a_m are given by

$$a_m = \int_0^1 f(x) \exp(-2\pi i m x)\, dx.$$

We again have convergence results:

Theorem 23. *Let $f\colon [0,1] \to \mathbb{C}$ be periodic.*

(a) The Fourier series converges uniformly provided f is continuously differentiable (this can be weakened to "absolutely continuous", but we won't need that level of generality).

(b) The Fourier series converges absolutely as long as $\sum_{m\in\mathbb{Z}} |a_m| < \infty$.

Example 24. If $f\colon \mathbb{R} \to \mathbb{C}$ is continuously differentiable with period 1, and α is an irrational number, then

$$\lim_{n\to\infty} \frac{f(\alpha) + \cdots + f(n\alpha)}{n} = \int_0^1 f(x)\, dx.$$

Proof. Just write $f(x) = \sum_m a_m \exp(2\pi i m x)$. Then note that for $m \neq 0$, if we let $z = \exp(2\pi i m\alpha)$ then

$$\frac{z^1 + z^2 + \cdots + z^n}{n} = \frac{z(1-z^n)}{n(1-z)} \to 0$$

as long as $z \neq 1$, which holds since z is not a root of unity. This leaves just the contribution form $a_0 = \int_0^1 f(x)\, dx$. □

In general, Fourier-type sums are good things to keep an eye out for, even if they don't explicitly come from Fourier series. For example, given a complex polynomial $p(z)$ (or even a series):

- The discrete sum $\sum_{k=0}^{n-1} p\left(e^{\frac{2\pi i k}{n}}\right)$ extracts the coefficients with indices divisible by n,

- the integral $\int_{t=0}^{2\pi} p(e^{it})\,dt = 2\pi \cdot p(0)$ extracts the constant term of the polynomial,

and so on. This is related to complex analysis, in which it turns complex differentiable functions $\mathbb{C} \to \mathbb{C}$ *are* exactly the same as complex analytic functions, which means you can go nuts with all sorts of beautiful results such as Cauchy's theorem.

5 Problems

1. Evaluate the improper integral
$$\int_0^1 \frac{\log(1-x)}{x}\,dx.$$

2. Determine the value of the improper integral
$$\int_0^\infty \frac{x}{e^x - 1}\,dx.$$

3. (a) Show that that $\min(a,b) = \int_0^\infty \mathbf{1}_{\le a}(t)\mathbf{1}_{\le b}(t)\,dt$ for any nonnegative real numbers $a, b \ge 0$. (What do you think $\mathbf{1}_{\le c}(t)$ means?)

   (b) Show that if $r_1, \dots, r_n$ are nonnegative reals and $x_1, \dots, x_n$ are real numbers then
$$\sum_{i=1}^n \sum_{j=1}^n \min(r_i, r_j) x_i x_j \ge 0.$$

4. For each continuous function $f\colon [0,1] \to \mathbb{R}$ let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(f) = \int_0^1 x f(x)^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

5. Compute
$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \dots + \frac{1}{\sqrt{4n^2 - n^2}} \right].$$

6. Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a,b]$ to $(0,\infty)$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ but $f \neq g$. For every positive integer n, define
$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n}\,dx.$$
Show that $I_1, I_2, I_3, \dots$ is an increasing sequence with $\lim_{n\to\infty} I_n = \infty$.

7. Let $a_0, a_1, \dots, a_n, x$ be real numbers, where $0 < x < 1$, satisfying
$$a_0 + \frac{a_1}{1+x} + \frac{a_2}{1+x+x^2} + \dots + \frac{a_n}{1+x+x^2+\dots+x^n} = 0.$$
Prove that for some $0 < y < 1$ we have
$$a_0 + a_1 y + a_2 y^2 + \dots + a_n y^n = 0.$$

8. Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \frac{a}{a^2+b^2}.$$

9. Evaluate the following:

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots \right) \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots \right) dx.$$

10. Show that

$$\int_0^1 x^{-x}\, dx = \sum_{n\geq 1} n^{-n}.$$

11. Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\, dx = 0$. Determine the largest possible value of

$$\int_1^3 \frac{f(x)}{x}\, dx.$$

12. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $f(x) \geq 1$ for all x. Suppose that

$$f(x)f(2x)\dots f(nx) \leq 2018n^{2019}$$

for every positive integer n and $x \in \mathbb{R}$. Must f be constant?

13. Show that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

14. A rectangle in $\mathbb{R}^2$ is called *great* if either its width or height is an integer. Prove that if a rectangle X can be dissected into great rectangles, then the rectangle X is itself great.

15. Compute

$$\sum_{k\geq 0} \frac{2^k}{5^{2^k}+1}.$$

16. Determine the value of

$$\lim_{n\to\infty} \left[\frac{2^{1/n}}{n+1} + \frac{2^{2/n}}{n+\frac{1}{2}} + \cdots + \frac{2^{n/n}}{n+\frac{1}{n}} \right].$$

17. For any continuous function $f\colon [0,1] \to \mathbb{R}$ let

$$\mu(f) = \int_0^1 f(x)\, dx, \quad \operatorname{Var}(f) = \int_0^1 (f(x) - \mu(f))^2\, dx, \quad M(f) = \max_{0\leq x\leq 1} |f(x)|.$$

Show that if $f, g\colon [0,1] \to \mathbb{R}$ are continuous functions then

$$\operatorname{Var}(fg) \leq 2\operatorname{Var}(f)M(g)^2 + 2\operatorname{Var}(g)M(f)^2.$$

18. For $m \geq 3$, a list of $\binom{m}{3}$ real numbers a_{ijk} (where $1 \leq i < j < k \leq m$) is said to be area definite for $\mathbb{R}^n$ if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of m points $A_1, \ldots, A_m$ in $\mathbb{R}^n$. For example, the list of four numbers $a_{123} = a_{124} = a_{134} = 1$, $a_{234} = -1$ is area definite for $\mathbb{R}^2$. Prove that if a list of $\binom{m}{3}$ numbers is area definite for $\mathbb{R}^2$, then it is area definite for $\mathbb{R}^3$.

19. Prove that

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} \binom{n}{k} \right)^{\frac{1}{n(n+1)}} = \sqrt{e}.$$

20. Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.$$

21. Let $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$. Prove that

$$\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx$$

diverges.

22. A rectangular prism X is contained within a rectangular prism Y.

    (a) Is it possible the surface area of X exceeds that of Y?

    (b) Is it possible the sum of the 12 side lengths of X exceeds that of Y?

23. For $a, b, c > 0$ prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{c+a} \geq \frac{12}{3a+b} + \frac{12}{3b+c} + \frac{12}{3c+a}.$$

24. Define a function $w\colon \mathbb{Z} \to \mathbb{Z}$ as follows. For $|a|, |b| \leq 2$, let $w(a,b)$ be as in the table shown; otherwise, let $w(a,b) = 0$.

	$w(a,b)$	b: -2	-1	0	1	2
a	-2	-1	-2	2	-2	-1
	-1	-2	4	-4	4	-2
	0	2	-4	12	-4	2
	1	-2	4	-4	4	-2
	2	-1	-2	2	-2	-1

For every finite nonempty subset S of $\mathbb{Z} \times \mathbb{Z}$, prove that

$$A(S) := \sum_{(\mathbf{s},\mathbf{s}') \in S \times S} w(\mathbf{s} - \mathbf{s}') > 0.$$

25. Evaluate
$$\lim_{x\to 1^-} \prod_{n\ge 0} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

26. Suppose that $f\colon [0,1]^2 \to \mathbb{R}$ is continuous. Show that
$$\begin{aligned}
&\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx \\
&\le \left(\int_0^1 \int_0^1 f(x,y)\, dx\, dy\right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\, dx\, dy.
\end{aligned}$$

27. For each positive integer k, let $A(k)$ be the number of odd divisors of k in the interval $\left[1, \sqrt{2k}\right)$. Evaluate:
$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{A(k)}{k}.$$