PROBLEMS ON RECURRENCES

1. Let $T_0 = 2, T_1 = 3, T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are: 2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392. Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

2. For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in simplest form.)

3. Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n. (Here, $[x]$ is the greatest integer $\leq x$.)

4. Define u_n by $u_0 = 0, u_1 = 4$, and $u_{n+2} = \frac{6}{5}u_{n+1} - u_n$. Show that $|u_n| \leq 5$ for all n. (In fact, $|u_n| < 5$ for all n. Can you show this?)

5. Show that the next integer above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

6. Let $a_0 = 0, a_1 = 1$, and for $n \geq 2$ let $a_n = 17a_{n-1} - 70a_{n-2}$. For $n > 6$, show that the first (most significant) digit of a_n (when written in base 10) is a 3.

7. Let a, b, c denote the (real) roots of the polynomial $P(t) = t^3 - 3t^2 - t + 1$. If $u_n = a^n + b^n + c^n$, what linear recursion is satisfied by $\{u_n\}$? If a is the largest of the three roots, what is the closest integer to a^5?

8. Solve the first order recursion given by $x_0 = 1$ and $x_n = 1 + (1/x_{n-1})$. Does $\{x_n\}$ approach a limiting value as n increases?

9. If $u_0 = 0, u_1 = 1$, and $u_{n+2} = 4(u_{n+1} - u_n)$, find u_{16}.

10. Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

11. Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{i=0}^{\infty}\left(1 - \frac{1}{a_k}\right)$$

in closed form.

12. (a) Define $u_0 = 1, u_1 = 1$, and for $n \geq 1$,

$$2u_{n+1} = \sum_{k=0}^{n}\binom{n}{k}u_k u_{n-k}.$$

Find a simple expression for $F(x) = \sum_{n \geq 0} u_n \frac{x^n}{n!}$. Express your answer in the form $G(x) + H(x)$, where $G(x)$ is even (i.e., $G(-x) = G(x)$) and $H(x)$ is odd (i.e., $H(-x) = -H(x)$).

(b) Define $u_0 = 1$ and for $n \geq 0$,

$$2u_{n+1} = \sum_{k=0}^{n} \binom{n}{k} u_k u_{n-k}.$$

Find a simple expression for u_n.

13. For a positive integer n and any real number c , define x_k recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix n and then take c to be the largest value for which $x_{n+1} = 0$. Find x_k in terms of n and k, $1 \leq k \leq n$.

14. Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

15. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \cdots \right\}.$$

16. Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

17. Let $a_1 < a_2$ be two given integers. For any integer $n \geq 3$, let a_n be the smallest integer which is larger than a_{n-1} and can be uniquely represented as $a_i + a_j$, where $1 \leq i < j \leq n-1$. Given that there are only a finite number of even numbers in $\{a_n\}$, prove that the sequence $\{a_{n+1} - a_n\}$ is eventually periodic, i.e. that there exist positive integers T, N such that for all integers $n > N$, we have

$$a_{T+n+1} - a_{T+n} = a_{n+1} - a_n.$$

18. Let k be an integer greater than 1. Suppose that $a_0 > 0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n > 0$. Evaluate

$$\lim_{n \to \infty} \frac{a_n^{k+1}}{n^k}.$$

19. Let $x_0 = 1$ and for $n \geq 0$, let $x_{n+1} = 3x_n + \lfloor x_n\sqrt{5} \rfloor$. In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}. ($\lfloor a \rfloor$ means the largest integer $\leq a$.)

20. (a) Let $a_0, \ldots, a_{k-1}$ be real numbers, and define

$$a_n = \frac{1}{k}(a_{n-1} + a_{n-2} + \cdots + a_{n-k}), \quad n \geq k.$$

Find $\lim_{n\to\infty} a_n$ (in terms of $a_0, a_1, \ldots, a_{k-1}$).

(b) Somewhat more generally, let $u_1, \ldots, u_k \geq 0$ with $\sum u_i = 1$ and $u_k \neq 0$. Assume that the polynomial $x^k - u_1x^{k-1} - u_2x^{k-2} - \cdots - u_k$ cannot be written in the form $P(x^d)$ for some polynomial P and some $d > 1$. Now define

$$a_n = u_1a_{n-1} + u_2a_{n-2} + \cdots + u_ka_{n-k}, \quad n \geq k.$$

Again find $\lim_{n\to\infty} a_n$. (Part (a) is the case $u_1 = \cdots = u_k = 1/k$.)

21. (a) (repeats Congruence and Divisibility Problem #22) Define u_n recursively by $u_0 = u_1 = u_2 = u_3 = 1$ and

$$u_nu_{n-4} = u_{n-1}u_{n-3} + u_{n-2}^2, \quad n \geq 4.$$

Show that u_n is an integer.

(b) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = 1$ and

$$u_nu_{n-5} = u_{n-1}u_{n-4} + u_{n-2}u_{n-3}, \quad n \geq 5.$$

(c) (much harder) Do the same for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = 1$ and

$$u_nu_{n-6} = u_{n-1}u_{n-5} + u_{n-2}u_{n-4} + u_{n-3}^2, \quad n \geq 6,$$

and for $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$ and

$$u_nu_{n-7} = u_{n-1}u_{n-6} + u_{n-2}u_{n-5} + u_{n-3}u_{n-4}, \quad n \geq 7.$$

(d) What about $u_0 = u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = u_7 = 1$ and

$$u_nu_{n-8} = u_{n-1}u_{n-7} + u_{n-2}u_{n-6} + u_{n-3}u_{n-5} + u_{n-4}^2, \quad n \geq 8?$$

22. (*very* difficult) Let $a_0, a_1, \ldots$ satisfy a homogeneous linear recurrence (of finite degree) with constant coefficients. I.e., for some complex (or real, if you prefer) numbers $\nu_1, \ldots, \nu_k$ we have

$$a_n = \nu_1a_{n-1} + \cdots + \nu_ka_{n-k}$$

for all $n \geq k$. Define

$$b_n = \begin{cases} 1, & a_n \neq 0 \\ 0, & a_n = 0. \end{cases}$$

Show that b_n is eventually periodic, i.e., there exists $p > 0$ such that $b_n = b_{n+p}$ for all n sufficiently large.

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