

## PROBLEMS ON INEQUALITIES

1. For  $p > 1$  and  $a_1, a_2, \dots, a_n$  positive, show that

$$\sum_{k=1}^n \left( \frac{a_1 + a_2 + \dots + a_k}{k} \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{k=1}^n a_k^p.$$

2. If  $a_n > 0$  for  $n = 1, 2, \dots$ , show that

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \dots a_n} \leq e \sum_{n=1}^{\infty} a_n,$$

provided that  $\sum_{n=1}^{\infty} a_n$  converges.

3. For  $n = 1, 2, 3, \dots$  let

$$x_n = \frac{1000^n}{n!}.$$

Find the largest term of the sequence.

4. Suppose that  $a_1, a_2, \dots, a_n$  with  $n \geq 2$  are real numbers greater than  $-1$ , and all the numbers  $a_j$  have the same sign. Show that

$$(1 + a_1)(1 + a_2) \dots (1 + a_n) > 1 + a_1 + a_2 + \dots + a_n.$$

5. If  $a_1, \dots, a_{n+1}$  are positive real numbers with  $a_1 = a_{n+1}$ , show that

$$\sum_{i=1}^n \left( \frac{a_i}{a_{i+1}} \right)^n \geq \sum_{i=1}^n \frac{a_{i+1}}{a_i}.$$

6. Show that for any real numbers  $a_1, a_2, \dots, a_n$ ,

$$\left( \sum_{i=1}^n \frac{a_i}{i} \right)^2 \leq \sum_{i=1}^n \sum_{j=1}^n \frac{a_i a_j}{i + j - 1}.$$

7. Let  $y = f(x)$  be a continuous, strictly increasing function of  $x$  for  $x \geq 0$ , with  $f(0) = 0$ , and let  $f^{-1}$  denote the inverse function to  $f$ . If  $a$  and  $b$  are nonnegative constants, then show that

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy.$$

8. Let  $a_1, a_2, \dots, a_n$  be real numbers. Show that

$$\min_{i < j} (a_i - a_j)^2 \leq M^2 (a_1^2 + \dots + a_n^2),$$

where

$$M^2 = \frac{12}{n(n^2 - 1)}.$$

9. Let  $f$  be a continuous function on the interval  $[0, 1]$  such that  $0 < m \leq f(x) \leq M$  for all  $x$  in  $[0, 1]$ . Show that

$$\left( \int_0^1 \frac{dx}{f(x)} \right) \left( \int_0^1 f(x) dx \right) \leq \frac{(m + M)^2}{4mM}.$$

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9. Let  $f$  be a continuous function on the interval  $[0, 1]$  such that  $0 < m \leq f(x) \leq M$  for all  $x$  in  $[0, 1]$ . Show that

$$\left( \int_0^1 \frac{dx}{f(x)} \right) \left( \int_0^1 f(x) dx \right) \leq \frac{(m + M)^2}{4mM}.$$

10. Consider any sequence  $a_1, a_2, \dots$  of real numbers. Show that

$$\sum_{n=1}^{\infty} a_n \leq \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left( \frac{r_n}{n} \right)^{1/2}$$

where

$$r_n = \sum_{k=n}^{\infty} a_k^2.$$

(If the left-hand side of the inequality is  $\infty$ , then so is the right-hand side.)

11. Show that

$$\frac{1}{(n-1)!} \int_n^{\infty} w(t) e^{-t} dt < \frac{1}{(e-1)^n},$$

where  $t$  is real,  $n$  is a positive integer, and

$$w(t) = (t-1)(t-2) \cdots (t-n+1).$$

12. Let  $(a_n)_{n=1}^{\infty}$  and  $(b_n)_{n=1}^{\infty}$  be two sequences of positive numbers. Show that the following statements are equivalent:

- There is a sequence  $(c_n)_{n=1}^{\infty}$  of positive numbers such that  $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$  and  $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$  both converge.
- $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$

13. Suppose that  $a, b, c$  are real numbers in the interval  $[-1, 1]$  such that  $1 + 2abc \geq a^2 + b^2 + c^2$ . Prove that  $1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}$  for all positive integers  $n$ .

14. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a two times differentiable function satisfying  $f(0) = 1, f'(0) = 0$  and for all  $x \in [0, \infty)$ , it satisfies

$$f''(x) - 5f'(x) + 6f(x) \geq 0$$

Prove that, for all  $x \in [0, \infty)$ ,

$$f(x) \geq 3e^{2x} - 2e^{3x}$$

15. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function satisfying  $xf(y) + yf(x) \leq 1$  for every  $x, y \in [0, 1]$ .  
 (a) Show that  $\int_0^1 f(x)dx \leq \frac{\pi}{4}$ .  
 (b) Find such a function for which equality occurs.

16. For what pairs of positive real numbers  $(a, b)$  does the improper integral shown converge?

$$\int_b^\infty \left( \sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

17. Let  $A$  be a positive real number. What are the possible values of  $\sum_{j=0}^{\infty} x_j^2$ , given that  $x_0, x_1, \dots$  are positive numbers for which  $\sum_{j=0}^{\infty} x_j = A$ ?

18. Let  $f(x)$  be a continuous real-valued function defined on the interval  $[0, 1]$ . Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| dx dy \geq \int_0^1 |f(x)| dx$$

19. For each continuous function  $f : [0, 1] \rightarrow \mathbb{R}$ , let  $I(f) = \int_0^1 x^2 f(x) dx$  and  $J(f) = \int_0^1 x (f(x))^2 dx$ . Find the maximum value of  $I(f) - J(f)$  over all such functions  $f$ .  
 20. Suppose that  $f : [0, 1] \rightarrow \mathbb{R}$  has a continuous derivative and that  $\int_0^1 f(x) dx = 0$ . Prove that for every  $\alpha \in (0, 1)$ ,

$$\left| \int_0^\alpha f(x) dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|$$

21. For  $m \geq 3$ , a list of  $\binom{m}{3}$  real numbers  $a_{ijk}$  ( $1 \leq i < j < k \leq m$ ) is said to be area definite for  $\mathbb{R}^n$  if the inequality

$$\sum_{1 \leq i < j < k \leq m} a_{ijk} \cdot \text{Area}(\triangle A_i A_j A_k) \geq 0$$

holds for every choice of  $m$  points  $A_1, \dots, A_m$  in  $\mathbb{R}^n$ . For example, the list of four numbers  $a_{123} = a_{124} = a_{134} = 1, a_{234} = -1$  is area definite for  $\mathbb{R}^2$ . Prove that if a list of  $\binom{m}{3}$  numbers is area definite for  $\mathbb{R}^2$ , then it is area definite for  $\mathbb{R}^3$ .

22. Let  $X_1, X_2, \dots$  be independent random variables with the same distribution, and let  $S_n = X_1 + X_2 + \dots + X_n, n = 1, 2, \dots$ . For what real numbers  $c$  is the following statement true:

$$\mathbb{P} \left( \left| \frac{S_{2n}}{2n} - c \right| \leq \left| \frac{S_n}{n} - c \right| \right) \geq \frac{1}{2}.$$

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