6.5.3 The Paley ETF

There is a simple construction of an ETF made of $2M$ vectors in M dimensions (corresponding to a $M \times 2M$ matrix) known as the Paley ETF that is essentially a partial Discrete Fourier Transform matrix. While we refer the reader to [BFMW13] for the details the construction consists of picking rows of the $p \times p$ Discrete Fourier Transform (with $p \cong 1 \mod 4$ a prime) with indices corresponding to quadratic residues modulo p. Just by coherence considerations this construction is known to be RIP for $s \approx \sqrt{p}$ but conjectured [BFMW13] to be RIP for $s \approx \frac{p}{\text{polylog} p}$, which would be predicted if the choice os rows was random (as discussed above)[30]. Although partial conditional (conditioned on a number theory conjecture) progress on this conjecture has been made [BMM14] no unconditional result is known for $s \ll \sqrt{p}$. This motivates the following Open Problem.

Open Problem 6.4 *Does the Paley Equiangular tight frame satisfy the Restricted Isometry Property pass the square root bottleneck? (even by logarithmic factors?).*

We note that [BMM14] shows that improving polynomially on this conjecture implies an improvement over the Paley clique number conjecture (Open Problem 8.4.)

References

[BMM14] A. S. Bandeira, D. G. Mixon, and J. Moreira. A conditional construction of restricted isometries. *Available online at arXiv:1410.6457 [math.FA]*, 2014.

[BFMW13] A. S. Bandeira, M. Fickus, D. G. Mixon, and P. Wong. The road to deterministic matrices with the restricted isometry property. *Journal of Fourier Analysis and Applications*, 19(6):1123–1149, 2013.

[30] We note that the quadratic residues are known to have pseudorandom properties, and indeed have been leveraged to reduce the randomness needed in certain RIP constructions [BFMM14]