18.S66 PROBLEMS #1

Spring 2003

All problems are supposed to be proved combinatorially, in most cases by exhibiting an explicit bijection between two sets. Try to give the most elegant proof possible. No credit for using recurrence relations, generating functions, etc.

The following notation is used throughout for certain sets of numbers:

$\mathbb{N}$	nonnegative integers
$\mathbb{P}$	positive integers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$[n]$	the set $\{1,2,\ldots,n\}$ when $n \in \mathbb{N}$

The symbol $(*)$ after a problem number means that the result of the problem is known, but a combinatorial proof is not known. Similarly the symbol (?) means that the result is known, but I am uncertain whether a combinatorial proof is known.

1. The number of subsets of an n-element set is 2^n.

2. A *composition* of n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers such that $\sum \alpha_i = n$. Show that the number of compositions of n is 2^{n-1}.

3. The total number of parts of all compositions of n is equal to $(n+1)2^{n-2}$.

4. For $n \geq 2$, the number of compositions of n with an even number of even parts is equal to 2^{n-2}.

5. Fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets S_i of $\{1,2,\ldots,n\}$ subject to each of the following conditions *separately*, i.e., the three parts are independent problems (all with the same general method of solution).

   (a) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$

(b) The S_i's are pairwise disjoint.

(c) $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$

6. If S is an n-element set, then let $\binom{S}{k}$ denote the set of all k-element subsets of S. Let $\binom{n}{k} = \#\binom{S}{k}$, the number of k-subsets of an n-set. (Thus we are *defining* the binomial coefficient $\binom{n}{k}$ combinatorially when $n, k \in \mathbb{N}$.) Show that

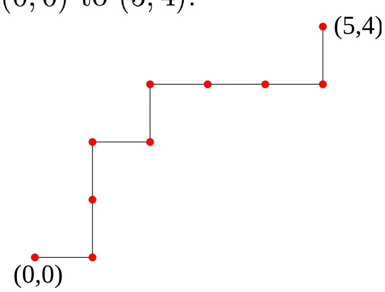
$$k!\binom{n}{k} = n(n-1)\cdots(n-k+1).$$

7. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. Here x and y are indeterminates and we define

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

NOTE. Both sides are polynomials in x and y. If two polynomials $P(x, y)$ and $Q(x, y)$ agree for $x, y \in \mathbb{N}$ then they agree as polynomials. Hence it suffices to assume $x, y \in \mathbb{N}$.

8. Let $m, n \geq 0$. How many lattice paths are there from $(0,0)$ to (m,n), if each step in the path is either $(1,0)$ or $(0,1)$? The figure below shows such a path from $(0,0)$ to $(5,4)$.



9. For $n \geq 0$, $2\binom{2n-1}{n} = \binom{2n}{n}$.

10. For $n \geq 1$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

11. For $n \geq 0$,

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}, \tag{1}$$

12. For $n \geq 0$,

$$\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}.$$

13. For $n \geq 0$,

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n.$$

14. Show that

$$\sum_{i=0}^{m} \binom{x+y+i}{i} \binom{y}{a-i} \binom{x}{b-i} = \binom{x+a}{b} \binom{y+b}{a},$$

where $m = \min(a, b)$.

15. For $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (x-1)^j.$$

16. (?) Fix $n \geq 0$. Then

$$\sum_{i+j+k=n} \binom{i+j}{i} \binom{j+k}{j} \binom{k+i}{k} = \sum_{r=0}^{n} \binom{2r}{r}.$$

Here $i, j, k \in \mathbb{N}$.

17. Show that if p is prime and $a \in \mathbb{P}$, then $a^p - a$ is divisible by p. (A combinatorial proof would consist of exhibiting a set S with $a^p - a$ elements and a partition of S into pairwise disjoint subsets, each with p elements.)

18. (a) Let p be a prime. Show that $\binom{2p}{p} - 2$ is divisible by p^2.

    (b) $(*)$ Show in fact that if $p > 3$, then $\binom{2p}{p} - 2$ is divisible by p^3.

19. A *multiset* M is, informally, a set with repeated elements, such as $\{1,1,1,2,4,4,4,5,5\}$, abbreviated $\{1^3,2,4^3,5^2\}$. The number of appearances of i in M is called the *multiplicity* of i, denoted $\nu_M(i)$ or just $\nu(i)$. The definition of a *submultiset* N of M should be clear, viz., $\nu_N(i) \le \nu_M(i)$ for all i. Let $M = \{1^{\nu_1}, 2^{\nu_2}, \ldots, k^{\nu_k}\}$. How many submultisets does M have?

20. The *size* or *cardinality* of a multiset M, denoted $\#M$ or $|M|$, is its number of elements, counting repetitions. For instance, if

$$M = \{1,1,1,2,4,4,4,5,5\}$$

then $\#M = 9$. A mulitset M is *on* a set S if every element of M is an element of S. Let $\left(\!\binom{n}{k}\!\right)$ denote the number of k-element multisets on an n-set, i.e., the number of ways of choosing, without regard to order, k elements from an n-element set if repetitions are allowed. Then

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

21. Fix $k, n \ge 0$. Find the number of solutions in nonnegative integers to

$$x_1 + x_2 + \cdots + x_k = n.$$

22. (*) Let $n \ge 2$ and $t \ge 0$. Let $f(n,t)$ be the number of sequences with n x's and $2t$ a_{ij}'s, where $1 \le i < j \le n$, such that each a_{ij} occurs between the ith x and the jth x in the sequence. (Thus the total number of terms in each sequence is $n + 2t\binom{n}{2}$.) Then

$$f(n,t) = \frac{(n+tn(n-1))!}{n!\, t!^n (2t)!^{\binom{n}{2}}} \prod_{j=1}^{n} \frac{((j-1)t)!^2 (jt)!}{(1+(n+j-2)t)!}.$$

23. The *Fibonacci numbers* F_n are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \ge 2$. The number $f(n)$ of compositions of n with parts 1 and 2 is F_{n+1}. (There is at this point no set whose cardinality is known to be F_{n+1}, so you should simply verify that $f(n)$ satisfies the Fibonacci recurrence and has the right initial values.)

24. The number of compositions of n with all parts > 1 is F_{n-1}.

25. The number of compositions of n with odd parts is F_n.

26. How many subsets S of $[n]$ don't contain two consecutive integers?

27. How many binary sequences (i.e., sequences of 0's and 1's) $(\varepsilon_1, \ldots, \varepsilon_n)$ satisfy

$$\varepsilon_1 \leq \varepsilon_2 \geq \varepsilon_3 \leq \varepsilon_4 \geq \varepsilon_5 \leq \cdots ?$$

28. Show that

$$\sum a_1 a_2 \cdots a_k = F_{2n},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

29. (?) Show that

$$\sum (2^{a_1 - 1} - 1) \cdots (2^{a_k - 1} - 1) = F_{2n-2},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

30. (?) Show that

$$\sum 2^{\{\# i \,:\, a_i = 1\}} = F_{2n+1},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

31. (?) The number of sequences $(\delta_1, \delta_2, \ldots, \delta_n)$ of 0's, 1's, and 2's such that 0 is never immediately followed by a 1 is equal to F_{2n+2}.

32. Show that $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$.

33. Show that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

34. Continuing Exercise 5, fix positive integers n and k. Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets S_i of $\{1, 2, \ldots, n\}$ satisfying

$$S_1 \subseteq S_2 \supseteq S_3 \subseteq S_4 \supseteq S_5 \subseteq \cdots.$$

(The symbols $\subseteq$ and $\supseteq$ alternate.)