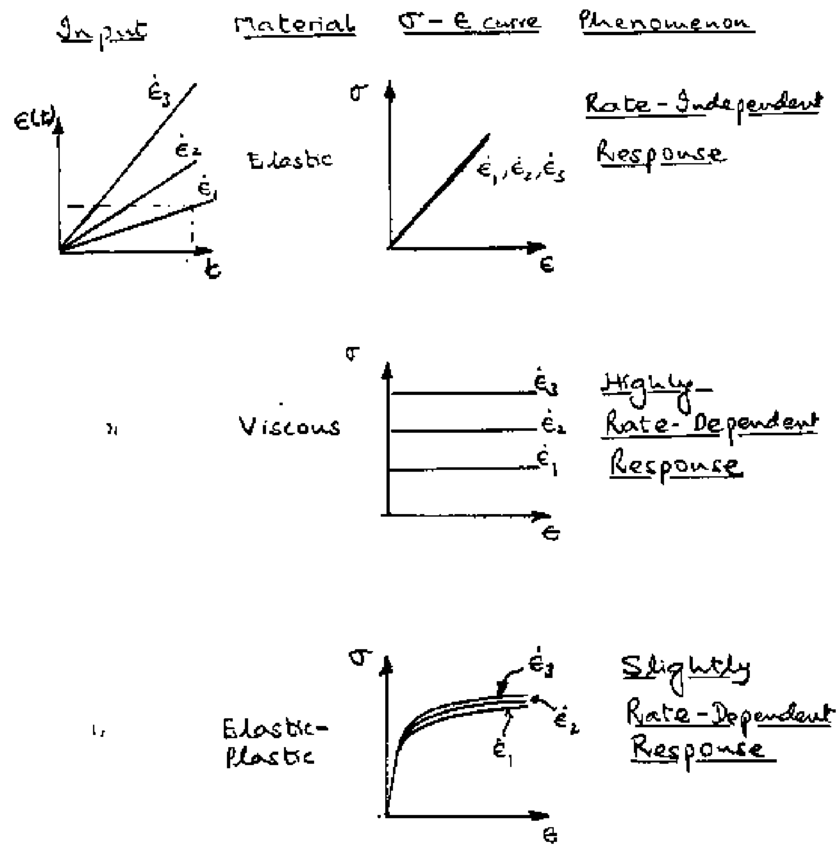
2.002 MECHANICS & MATERIALS II

INTRODUCTION TO THE MACROSCOPIC THEORY OF PLASTICITY

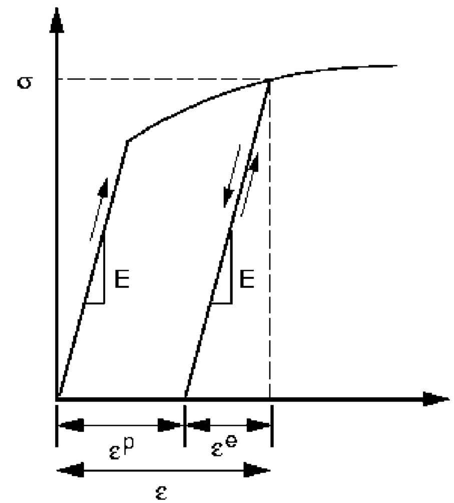
RATE-DEPENDENCE AND RATE-INDEPENDENCE



- Plastic deformation in metals is **thermally-activated** and **inherently rate-dependent**.

- However, the plastic stress-strain response of most single and polycrystalline materials at absolute temperatures $T < (1/3)T_m$, where T_m is the melting temperature of the material in degrees absolute, is only slightly rate-sensitive, and in this temperature regime it may be modeled as **rate-independent**.

Material	Melting Temp, C	T_m, K	$(1/3)T_m$, K	$\equiv$ C
Ti	1668	1941	647	374
Fe	1536	1809	603	330
Cu	1083	1356	452	179
Al	660	933	311	38
Pb	327	660	200	-73



The governing variables in the one–dimensional rate–independent constitutive model for elastic-plastic solids are

σ	Stress,
ϵ	Strain,
ϵ^p	Plastic strain,
$s > 0$	Plastic deformation resistance, internal variable with dimensions of stress. Initial value of s: $s_0 \equiv \sigma_y$ – yield strength in tension

The constitutive model consists of the following set of equations:

1. **Elastic strain**:

$$\epsilon^e = \epsilon - \epsilon^p$$

2. **Constitutive Equation For** σ:

$$\sigma = E\left[\epsilon^e\right] = E\left[\epsilon - \epsilon^p\right], \quad \text{E − Young's Modulus}$$

3. **Yield Condition**:

Let s denote an internal variable which is a nonzero, positive-valued scalar with the dimensions of stress. We call s the **deformation resistance**.

The assumption that only a single scalar characterizes the complex internal characteristics of a material is, of course, a gross simplification, but nevertheless, it is widely used to great effectiveness in engineering practice.

Next, we introduce a scalar valued function

$$f(\sigma, s) = |\sigma| - s,$$

called the **yield function**, and constrain the admissible states (σ, s) such that

$$\boxed{f(\sigma, s) = |\sigma| - s \leq 0.}$$

This is called the **yield condition**.

The set of values of $\{\sigma\}$ giving resulting in $f = 0$ for a given s is called the **yield surface**. In the present one-dimensional context the yield surface is the pair of points $(\sigma = -s, \sigma = +s)$. The yield surface defines the boundary of the elastic domain at a given s.

Note that a state (σ, s) with a value $f(\sigma, s) > 0$ is not admissible. For later use we note that when $f = 0$, this imposes the restriction that $\dot{f} \leq 0$.

To see this, let $f(t) = f(\sigma(t), s(t))$ be the value of f at time t, and consider a time $\tau = t + \Delta t$ with $\Delta t > 0$, then

$$f(\tau) = f(t) + \dot{f}(t)\,\Delta t + o(\Delta t).$$

Since $f(t) = 0$, we must have $\dot{f}(t) \leq 0$, otherwise $f(\tau) > 0$, which is inadmissible.

4. **Evolution Equation For ϵ^p, Flow Rule**:

The plastic strain is taken to evolve according to the flow rule

$$\dot{\epsilon}^p = \dot{\bar{\epsilon}}^p \, \mathrm{sign}(\sigma),$$

$$\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if } f < 0 \quad \text{— elastic}, \\ 0 & \text{if } f = 0, \quad \text{and} \quad \dot{f} < 0 \quad \text{— elastic unloading}, \\ > 0 & \text{if } f = 0, \quad \text{and} \quad \dot{f} = 0 \text{— plastic loading}, \end{cases}$$

$$\mathrm{sign}(\sigma) = \begin{cases} 1 & \text{if } \sigma > 0, \\ -1 & \text{if } \sigma < 0, \end{cases}$$

where $\dot{\bar{\epsilon}}^p$ is the **magnitude** of the plastic strain rate, and $\mathrm{sign}(\sigma)$ gives the **direction of plastic flow**.

The conditions for $\dot{\bar{\epsilon}}^p \geq 0$ are called the **loading/unloading conditions**.

$$
\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if } f < 0 \quad \text{— elastic,} \\ 0 & \text{if } f = 0, \quad \text{and} \quad \dot{f} < 0 \quad \text{— elastic unloading,} \\ > 0 & \text{if } f = 0, \quad \text{and} \quad \dot{f} = 0 \text{— plastic loading} \end{cases}
$$

If $f < 0$ then $\dot{\bar{\epsilon}}^p = 0$, the instantaneous response is **elastic**. If $f = 0$, then we have an **plastic-state** and it is possible that $\dot{\bar{\epsilon}}^p \geq 0$. If $f = 0$ and $\dot{f} < 0$, resulting in $\dot{\bar{\epsilon}}^p = 0$, we have **elastic unloading** from a plastic state. Finally, if $f = 0$ and $\dot{f} = 0$, resulting in $\dot{\bar{\epsilon}}^p > 0$, we have **plastic loading**.

Since $\dot{\bar{\epsilon}}^p = 0$ if $f < 0$, and $\dot{\bar{\epsilon}}^p > 0$ is possible only if $f = 0$, it follows that $\dot{\bar{\epsilon}}^p f = 0$:

$$
f \leq 0, \quad \dot{\bar{\epsilon}}^p \geq 0, \quad \dot{\bar{\epsilon}}^p f = 0.
$$

5. **Evolution Equation For s, Hardening Rule**:

Next, the evolution equation for the deformation resistance s is taken as

$$\dot{s} = h\,\dot{\bar{\epsilon}}^p, \quad h = \hat{h}(s)$$

where $\hat{h}$ is a **hardening function**.

The material is said to be **strain-hardening**, **perfectly plastic**, or **strain-softening** according as $h > 0$, $h = 0$ or $h < 0$, respectively.

6. **Consistency Condition**:

Since from a plastic state $f = 0$, $\dot{\bar{\epsilon}}^p = 0$ if $\dot{f} < 0$, and $\dot{\bar{\epsilon}}^p > 0$ is possible only if $\dot{f} = 0$, we have

$$\dot{\bar{\epsilon}}^p \dot{f} = 0 \quad \text{if} \quad f = 0.$$

This is called the **consistency (persistency) condition**. To elaborate, in order for $\dot{\bar{\epsilon}}^p > 0$, a state (σ, s) on the boundary of the elastic domain, that is one satisfying $f(\sigma, s) = 0$, must persist on the boundary of the elastic domain, so that $\dot{f}(\sigma, s) = 0$. That is, during a plastic process the pair (σ, s) must continue to satisfy the yield condition $f = |\sigma| - s = 0$. This is feasible only if

$$\dot{f} = \dot{\overline{|\sigma|}} - \dot{s} = 0$$

is satisfied.

7. **Magnitude of $\dot{\bar{\epsilon}}^p$. Alternate form for the Loading/Unloading Conditions**:

The consistency condition serves to determine the magnitude of the plastic strain rate, $\dot{\bar{\epsilon}}^p$, when plastic flow occurs.

Since

$$\frac{\mathrm{d}\;|\sigma|}{\mathrm{d}\sigma} = \mathrm{sign}(\sigma),$$

we have

$$\dot{\overline{|\sigma|}} = \mathrm{sign}(\sigma)\,\dot{\sigma}.$$

Hence, using the rate form of the constitutive equation for stress,

$$\dot{\sigma} = E\left[\dot{\epsilon} - \dot{\bar{\epsilon}}^p\,\mathrm{sign}(\sigma)\right]$$

and the evolution equation for s,

$$\dot{s} = h\,\dot{\bar{\epsilon}}^p,$$

we have

$$\begin{aligned}
\dot{f} &= \dot{\overline{|\sigma|}} - \dot{s}, \\
&= \mathsf{sign}(\sigma)\,\dot{\sigma} - \dot{s}, \\
&= \mathsf{sign}(\sigma)\,E\left[\dot{\epsilon} - \dot{\bar{\epsilon}}^p\,\mathsf{sign}(\sigma)\right] - h\,\dot{\bar{\epsilon}}^p, \\
&= \mathsf{sign}(\sigma)\,E\,[\dot{\epsilon}] - \dot{\bar{\epsilon}}^p\,\{E + h\} \leq 0.
\end{aligned}$$

We assume that

$$g \equiv \{E + h\} > 0.$$

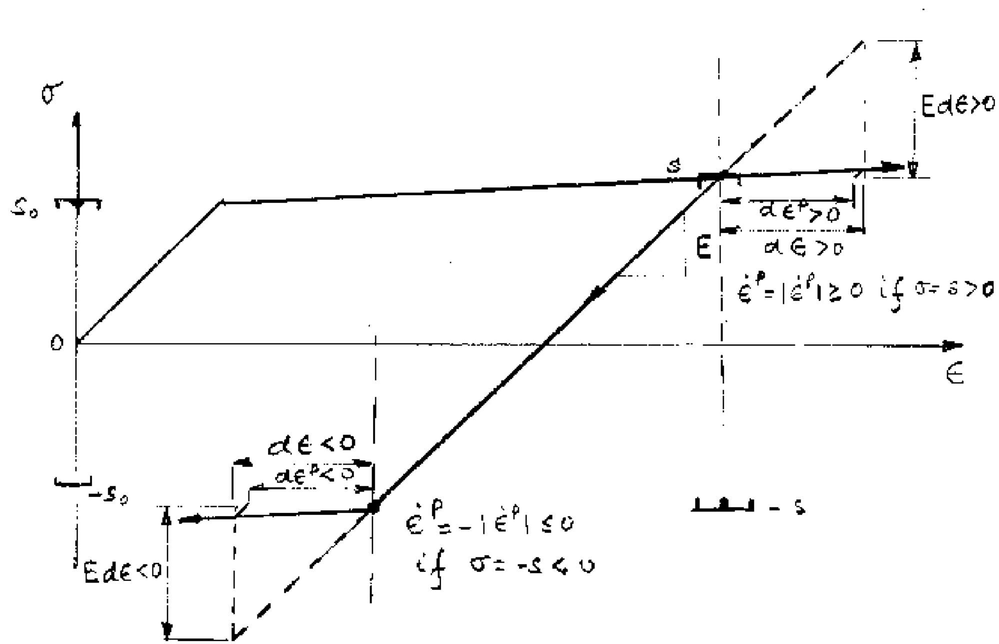
This is an important assumption; it sets a limit on the negative values of the strain hardening function h.

Thus the magnitude of the plastic strain rate is

$$\dot{\epsilon}^p = \begin{cases} 0 & \text{if } f < 0 \text{ elastic}, \\ 0 & \text{if } f = 0, \text{ and } \{\text{sign}(\sigma)\, E\, [\dot{\epsilon}]\} < 0, \\ g^{-1}\{\text{sign}(\sigma)\, E\, [\dot{\epsilon}]\} & \text{if } f = 0, \text{ and } \{\text{sign}(\sigma)\, E\, [\dot{\epsilon}]\} > 0, \end{cases}$$

where

$$g \equiv \{E + h\} > 0.$$



Schematic of plastic loading and unloading from a state of stress which satisfies the yield condition $f = |\sigma| - s = 0$.

8. **Elastic-plastic Tangent Moduli**:

During plastic loading

$$\begin{aligned}
\dot{\sigma} &= E\left[\dot{\epsilon} - \dot{\epsilon}^p\right] = E\left[\dot{\epsilon} - \dot{\bar{\epsilon}}^p \,\mathrm{sign}(\sigma)\right], \\
&= E\left[\dot{\epsilon} - g^{-1}\, E\left[\dot{\epsilon}\right]\right] = E\left[\dot{\epsilon} - \frac{E}{E+h}\,\dot{\epsilon}\right] = E\left[1 - \frac{E}{E+h}\right]\dot{\epsilon}. \\
&= \left(\frac{Eh}{E+h}\right)\dot{\epsilon},
\end{aligned}$$

Hence,

$$\dot{\sigma} = E^{ep}\left[\dot{\epsilon}\right],$$

with

$$E^{ep} = \begin{cases} E & \text{if } \dot{\bar{\epsilon}}^p = 0, \\ \left(\dfrac{Eh}{E+h}\right) & \text{if } \dot{\bar{\epsilon}}^p > 0, \end{cases}$$

is the the **elastic-plastic tangent modulus**.

This provides an interpretation of our assumption $g = E + h > 0$. During plastic loading, for a hardening material $h > 0$ and $E^{ep} > 0$. For a non-hardening material $h = 0$ and $E^{ep} = 0$. For a strain-softening material material $h < 0$ and $E^{ep} < 0$, but our assumption $g = E + h > 0$ precludes $E^{ep} = -\infty$.

SUMMARY OF 1-D FORMULATION

Regarding ϵ as the independent variable and $\{\sigma, \epsilon^p, s\}$ as the dependent variables, the one-dimensional rate-independent constitutive model for elastic-plastic solids with isotropic hardening consists of the following set of equations:

1. **Elastic strain**:

$$\epsilon^e = \epsilon - \epsilon^p.$$

2. **Constitutive Equation For** σ:

$$\sigma = E\left[\epsilon - \epsilon^p\right].$$

3. **Yield Condition**:

$$f = |\sigma| - s \leq 0.$$

4. **Flow Rule and Hardening Rule**:

$$\dot{\epsilon}^p = \dot{\bar{\epsilon}}^p \text{sign}(\sigma),$$
$$\dot{s} = h\,\dot{\bar{\epsilon}}^p, \quad h = \hat{h}(s).$$

5. **Complementarity Conditions and Consistency Condition**:

$$f \leq 0, \quad \dot{\bar{\epsilon}}^p \geq 0, \quad \dot{\bar{\epsilon}}^p f = 0.$$
$$\dot{\bar{\epsilon}}^p \dot{f} = 0 \quad \text{if} \quad f = 0.$$

6. **Magnitude of the plastic strain rate**:

$$\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if } f < 0, \\ 0 & \text{if } f = 0, \quad \text{and} \quad \{\text{sign}(\sigma)\, E\, [\dot{\epsilon}]\} < 0, \\ g^{-1}\, \{\text{sign}(\sigma)\, E\, [\dot{\epsilon}]\} & \text{if } f = 0, \quad \text{and} \quad \{\text{sign}(\sigma)\, E\, [\dot{\epsilon}]\} > 0, \end{cases}$$

with

$$g \equiv \{E + h\} > 0.$$

To complete this constitutive model for a given material, the material properties/functions that need to be determined are

1. The Young's modulus E.

2. The initial values s_0 of s. This is widely called the **yield strength** of the material and denoted by

$$\boxed{\sigma_y \equiv s_0.}$$

3. The strain-hardening function

$$\boxed{h = \widehat{h}(s)}.$$

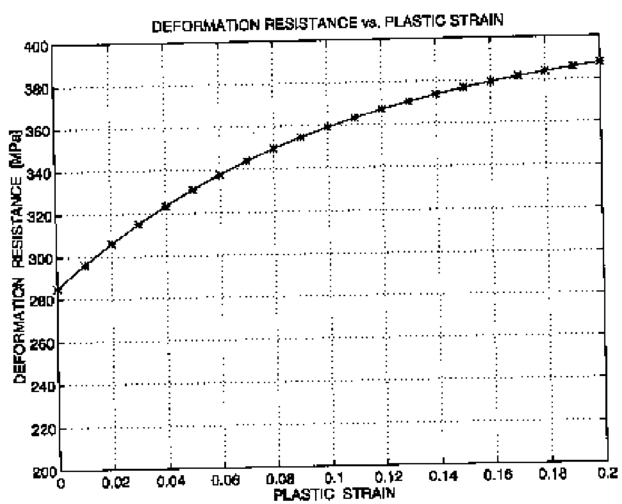
The hardening function $\widehat{h}(s)$ is determined as follows:

- Assume that E and the true stress-strain data (σ versus ϵ) have been obtained from a compression or a tension test.

- Then using $\epsilon^p = \epsilon - (\sigma/E)$ the (σ versus ϵ) data is converted into (σ versus ϵ^p).

  If the data is obtained from a compression test, then convert the data into ($|\sigma|$ versus $|\epsilon^p|$).

- Next, since $|\sigma| = s$ and $|\epsilon^p| = \bar{\epsilon}^p$ during plastic flow, the ($|\sigma|$ versus $|\epsilon^p|$) data is **identical** to (s versus $\bar{\epsilon}^p$) data, from which the desired hardening function can be determined as the slope $\left(h = \dfrac{\mathrm{d}s}{\mathrm{d}\bar{\epsilon}^p}\right.$ versus $\left.s\right)$.



s vs $\bar{\epsilon}^p$ for 6061-T6 al alloy at room temp.

3-Dimensional Theory

The governing variables in the three-dimensional theory are

σ_{ij}	Stress
$\epsilon_{ij} = (1/2)\left[\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right]$	Strain
ϵ_{ij}^p	Plastic strain
s	Isotropic deformation resistance, dimensions of stress, $s > 0$

The constitutive model consists of the following set of equations:

Elastic strain:

$$\epsilon^e_{ij} = \epsilon_{ij} - \epsilon^p_{ij}$$

Constitutive Equation For Stress:

$$\sigma_{ij} = \sum_{k,l} \mathcal{C}_{ijkl} \left[\epsilon_{kl} - \epsilon^p_{kl} \right].$$

$$\mathcal{C}_{ijkl} = \frac{E}{2(1+\nu)} \left\{ \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right\} + \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl},$$

E — Young's modulus,

ν — Poisson's ratio.

Yield Condition:

We introduce a **yield condition** which bounds the levels of stresses in the material. For isotropic materials, a simple yield condition is

$$f(\boldsymbol{\sigma}, s) \leq 0,$$

where $f(\boldsymbol{\sigma}, s)$ is a scalar-valued function of the applied stress $\boldsymbol{\sigma}$, and the scalar s is a **material property** called the **deformation resistance** of the material.

Isotropy requires that the dependence on $\boldsymbol{\sigma}$ in the function $f(\boldsymbol{\sigma}, s)$ can only appear in terms of its principal **invariants** $\{I_1, I_2, I_3\}$.

Since $\boldsymbol{\sigma}$ is symmetric, then so also is the **stress deviator**

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - (1/3)(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{1}.$$

The symmetric tensor $\boldsymbol{\sigma}'$ has only five independent components, and only two independent non-zero invariants:

$$J_2 = \frac{1}{2}\left[\sum_{i,j} \sigma'_{ij}\sigma'_{ij}\right], \qquad J_3 = \det \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix}.$$

Thus, instead of the list of invariants $\{I_1, I_2, I_3\}$ for the stress, we may use the following as an alternative list of invariants for $\boldsymbol{\sigma}$:

$$I_1 = \sum_k \sigma_{kk}, \qquad J_2 = \frac{1}{2}\left[\sum_{i,j} \sigma'_{ij}\sigma'_{ij}\right], \qquad J_3 = \det\left[\boldsymbol{\sigma}'\right].$$

Additional invariants may be defined in terms of $\{J_1, J_2, J_3\}$. The following list of invariants for the stress tensor are widely used in the theory of isotropic plasticity:

$$\bar{p} = -\frac{1}{3}I_1 \qquad \bar{\sigma} = \sqrt{\frac{3}{2}\sum_{i,j}\sigma'_{ij}\sigma'_{ij}}$$

The invariant $\bar{p}$ is called the **mean normal stress**, and $\bar{\sigma}$ is called the **equivalent tensile stress**.

The invariants $\bar{p}$ and $\bar{\sigma}$ written out in full take the forms:

1. **Mean normal stress**:

$$\bar{p} = -\frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right).$$

   In the case of a state of hydrostatic pressure, $\sigma_{ij} = -p\,\delta_{ij}$, the mean normal stress is $\bar{p} = p$.

2. **Equivalent tensile stress**:

$$\bar{\sigma} = \left|\left[\frac{1}{2}\left\{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right\} + 3\left\{\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right\}\right]^{1/2}\right|$$

   In the case of pure tension $\sigma_{11} = \sigma$, all other $\sigma_{ij} = 0$, the equivalent tensile stress is $\bar{\sigma} = |\sigma|$.

Thus we may express our isotropic yield condition as

$$\boxed{f(\bar{p}, \bar{\sigma}, s) \leq 0.}$$

For ductile metallic polycrystalline materials it has been found experimentally that the function $f(\bar{p}, \bar{\sigma}, s)$ can, to a very good approximation, be taken to be **independent** of $\bar{p}$, and the most widely used yield condition is the **Mises yield condition** propsed by by Richard von Mises in 1913:

$$\boxed{f(\boldsymbol{\sigma}, s) = \bar{\sigma} - s \leq 0,}$$

with

$$\bar{\sigma} = \left| \left[\frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} + 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right]^{1/2} \right| .$$

Note that for this yield function

$$f(\boldsymbol{\sigma}, s) = \bar{\sigma} - s = \sqrt{\frac{3}{2}\sum_{k,l} \boldsymbol{\sigma}'_{kl}\boldsymbol{\sigma}'_{kl}} - s$$

the components of **the outward normal to the yield surface**, $f = \bar{\sigma} - s = 0$, at the current stress point are

$$\frac{\partial f}{\partial \sigma_{ij}} = \left\{\frac{3}{2}\frac{\sigma'_{ij}}{\bar{\sigma}}\right\}.$$

Normality Flow Rule (Levy, Saint Venant):

$$\boxed{\dot{\epsilon}^p_{ij} = \dot{\bar{\epsilon}}^p \left(\frac{3}{2} \frac{\sigma'_{ij}}{\bar{\sigma}} \right)}$$

The quantity

$$\boxed{\dot{\bar{\epsilon}}^p \equiv \sqrt{\frac{2}{3} \sum_{i,j} \dot{\epsilon}^p_{ij} \dot{\epsilon}^p_{ij}} \geq 0.}$$

is called the **equivalent tensile plastic strain rate**.

Note that since σ'_{ij} is deviatoric, $\sum_{j=1}^{3} \dot{\epsilon}^p_{jj} = 0$. Thus, according to this flow rule, **plastic flow is incompressible**.

Equivalent tensile plastic strain rate:

$$\dot{\bar{\epsilon}}^p \equiv \sqrt{\frac{2}{3}\sum_{i,j}\dot{\epsilon}^p_{ij}\dot{\epsilon}^p_{ij}} = \left|\left[\frac{2}{9}\left\{(\dot{\epsilon}^p_{11}-\dot{\epsilon}^p_{22})^2+(\dot{\epsilon}^p_{22}-\dot{\epsilon}^p_{33})^2+(\dot{\epsilon}^p_{33}-\dot{\epsilon}^p_{11})^2\right\}\right.\right.$$
$$\left.\left.+\frac{4}{3}\left\{(\dot{\epsilon}^p_{12})^2+(\dot{\epsilon}^p_{23})^2+(\dot{\epsilon}^p_{31})^2\right\}\right]^{1/2}\right|$$

Let $\dot{\epsilon}^p_{11}$ denote the plastic strain rate in a simple tension/compression test. Then because of isotropy $\dot{\epsilon}^p_{22} = \dot{\epsilon}^p_{33}$, and becuase of plastic incompressibility

$$\dot{\epsilon}^p_{11}+\dot{\epsilon}^p_{22}+\dot{\epsilon}^p_{33}=0 \Longrightarrow \dot{\epsilon}^p_{22}=\dot{\epsilon}^p_{33}=-\left(\frac{1}{2}\right)\dot{\epsilon}^p_{11},$$

and all other $\dot{\epsilon}^p_{kl}=0$. Under these conditions

$$\dot{\bar{\epsilon}}^p = \left|\dot{\epsilon}^p_{11}\right|,$$

and hence the name equivalent tensile plastic strain rate.

The quantity

$$\boxed{\overline{\epsilon}^p(t) = \int_0^t \dot{\overline{\epsilon}}^p(\xi)\,\mathrm{d}\xi,}$$

is called the **equivalent tensile plastic strain**.

Hardening Rule:

$$\dot{s} = h\,\dot{\overline{\epsilon}}^p, \quad h = \widehat{h}(s) \text{ hardening function}$$

Complementarity Conditions and Consistency Condition:

$$f \le 0, \quad \dot{\overline{\epsilon}}^p \ge 0, \quad \dot{\overline{\epsilon}}^p f = 0,$$
$$\dot{\overline{\epsilon}}^p \dot{f} = 0 \quad \text{if} \quad f = 0.$$

Magnitude of the plastic strain rate:

$$\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if } f < 0, \\ 0 & \text{if } f = 0 \text{ and } \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\} < 0, \\ 0 & \text{if } f = 0 \text{ and } \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\} = 0, \\ \sqrt{\frac{3}{2}} g^{-1} \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\}, & \text{if } f = 0 \text{ and } \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\} > 0, \end{cases}$$

with

$$g \equiv \left[\frac{3E}{2(1+\nu)} + h\right] > 0,$$

$$n_{pq} \equiv \sqrt{\frac{3}{2}}\left(\frac{\sigma'_{pq}}{\bar{\sigma}}\right) \text{ outward unit normal to yield surface,}$$

$$\dot{\sigma}_{pq}^{\text{trial}} \equiv \sum_{r,s} \mathcal{C}_{pqrs}\dot{\epsilon}_{rs} \text{ trial stress rate.}$$

Summary

Strain Rate in Terms of Stress Rate:

$$\dot{\epsilon}_{ij} = \frac{1}{E}\left[(1+\nu)\dot{\sigma}_{ij} - \nu\left(\sum_k \dot{\sigma}_{kk}\right)\delta_{ij}\right] + \sqrt{\frac{3}{2}}\,\dot{\bar{\epsilon}}^p\, n_{ij}$$

$$\dot{\bar{\epsilon}}^p = \begin{cases} 0 & \text{if } f < 0, \\ 0 & \text{if } f = 0 \text{ and } \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\} < 0, \\ 0 & \text{if } f = 0 \text{ and } \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\} = 0, \\ \sqrt{\frac{3}{2}} g^{-1} \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\}, & \text{if } f = 0 \text{ and } \left\{\sum_{p,q} n_{pq}\dot{\sigma}_{pq}^{\text{trial}}\right\} > 0, \end{cases}$$

with **Mises yield condition**

$$f(\boldsymbol{\sigma}, s) = \bar{\sigma} - s \leq 0$$

$$\bar{\sigma} = \left| \left[\frac{1}{2} \left\{ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right\} + 3 \left\{ \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right\} \right]^{1/2} \right|,$$

and

$$g \equiv \left[\frac{3E}{2(1+\nu)} + h \right] > 0,$$

$$n_{pq} \equiv \sqrt{\frac{3}{2}} \left(\frac{\sigma'_{pq}}{\bar{\sigma}} \right) \text{ outward unit normal to yield surface,}$$

$$\dot{\sigma}_{pq}^{\text{trial}} \equiv \sum_{r,s} \mathcal{C}_{pqrs} \dot{\epsilon}_{rs} \text{ trial stress rate.}$$

Stress Rate in Terms of Strain Rate:

$$\dot{\sigma}_{ij} = \sum_{k,l} \mathcal{L}_{ijkl} \dot{\epsilon}_{kl}$$

$$\mathcal{L}_{ijkl} = \begin{cases} \mathcal{C}_{ijkl} & \text{if } \dot{\bar{\epsilon}}^p = 0, \\ \mathcal{C}_{ijkl} - (3/2)\, g^{-1} \left(m_{ij} m_{kl}\right) & \text{if } \dot{\bar{\epsilon}}^p > 0, \end{cases}$$

$$\mathcal{C}_{ijkl} = \frac{E}{2(1+\nu)} \left\{\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right\} + \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij}\delta_{kl},$$

$$g = \left[\frac{3E}{2(1+\nu)} + h\right] > 0,$$

$$m_{ij} \equiv \sum_{k,l} \mathcal{C}_{ijkl} n_{kl}, \qquad n_{ij} \equiv \sqrt{\frac{3}{2}} \left(\frac{\sigma'_{ij}}{\bar{\sigma}}\right)$$

$\mathcal{L}_{ijkl}$ are the **elasto-plastic tangent moduli**, and $\mathcal{C}_{ijkl}$ are the **elastic moduli**.

Hardening Rule:

$$\dot{s} = h\,\dot{\bar{\epsilon}}^p, \quad h = \hat{h}(s) \text{ hardening function}$$

This is a generalization of the one-dimensional case with $\dot{\bar{\epsilon}}^p$ the **equivalent tensile plastic strain rate**.

The quantity

$$\boxed{\bar{\epsilon}^p(t) = \int_0^t \dot{\bar{\epsilon}}^p(\xi)\,\mathrm{d}\xi,}$$

is the **equivalent tensile plastic strain**.

For monotonic proportional loading the evolution equation for s may be integrated to give s as a function of $\bar{\epsilon}^p$:

$$s = \hat{s}(\bar{\epsilon}^p).$$

To complete this constitutive model for a given material, the material properties/functions that need to be determined are the **elastic moduli**

$$(E, \nu)\,,$$

the **initial value**

$$s_0 \equiv \sigma_y \quad \textbf{yield strength}$$

of the deformation resistance s, and the **hardening function**

$$h = \hat{h}(s).$$

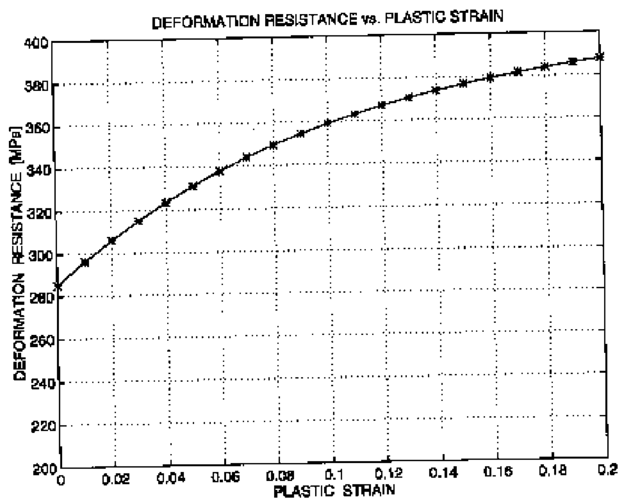
The hardening function $\widehat{h}(s)$ is determined from a **a simple tension/compression** test as follows:

- Assume that E and the true stress-strain data (σ versus ϵ) have been obtained from a compression or a tension test.

- Then using $\epsilon^p = \epsilon - (\sigma/E)$ the (σ versus ϵ) data is converted into (σ versus ϵ^p).

  If the data is obtained from a compression test, then convert the data into ($|\sigma|$ versus $|\epsilon^p|$).

- Next, since $|\sigma| = s$ and $|\epsilon^p| = \bar{\epsilon}^p$ during plastic flow, the ($|\sigma|$ versus $|\epsilon^p|$) data is **identical** to (s versus $\bar{\epsilon}^p$) data, from which the desired hardening function can be determined as the slope ($h = \dfrac{\mathrm{d}s}{\mathrm{d}\bar{\epsilon}^p}$ versus s) .



s vs $\bar{\epsilon}^p$ for 6061-T6 al alloy at room temp.

Concluding Remarks

- The equations for elastic-plastic deformation are coupled differential evolutions for the stress σ_{ij} and the deformation resistance s.

- The solution of complex boundary-value problems using these equations is best carried out numerically.

- The constitutive equations described here (with some possible change in notation and terminology) are the ones most widely used in modern commercial finite-element programs.