Troubles at the Origin: Consistent Usage and Properties of the Unilateral Laplace Transform

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The Laplace transform is a standard tool associated with the analysis of signals, models, and control systems, and is consequently taught in some form to almost all engineering students. The bilateral and unilateral forms of the Laplace transform are closely related, but have somewhat different domains of application. The bilateral transform is most frequently seen in the context of signal processing, whereas the unilateral transform is most often associated with the study of dynamic system response where the role of initial conditions takes on greater significance. In our teaching we have found some significant pitfalls associated with teaching our students to understand and apply the Laplace transform. These confusions extend to the presentation of this material in many of the available mathematics and engineering textbooks as well.

The most significant confusion in much of the textbook literature is how to deal with the origin in the application of the unilateral Laplace transform. That is, many texts present the transform of a time function $f(t)$ as

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{1}$$

without properly specifiying the meaning of the lower limit of integration. Said informally, does the integral include the origin fully, partially, or not at all? This issue becomes significant as soon as singularity functions such as the unit impulse are introduced. While it is not possible to devote full attention to this issue within the context of a typical undergraduate course, this "skeleton in the closet" as Kailath [8] called it needs to be brought out fully into the light. Our purpose in writing this article is to put forward a consistent set of Laplace transform definitions and properties which allow the correct analysis of dynamic systems in the presence of arbitrary initial conditions and where the system is driven by functions which include singularities. We also present reasonable mathematical support for these properties, as well as a consistent treatment of singularity functions, without becoming fully enmeshed in the theory of generalized functions, which quickly becomes too far divorced from applications.

To properly learn about and apply the unilateral Laplace transform, students need to be taught a consistent set of properties that correctly handle problems with arbitrary inputs and initial conditions. The proper form of the unilateral Laplace transform fully includes the origin

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt \tag{2}$$

as indicated by the 0^- notation. Thus the integral includes interesting events which happen at $t = 0$, such as impulses or higher-order singularity functions, steps, and the beginnings of other transients. Some texts refer to this form as the $\mathcal{L}_-$ transform. However, since we regard this as the only correct usage of the unilateral transform in the context of dynamic systems, we will omit any additional notation, and use the symbol $\mathcal{L}$ to represent the transform as defined in (2).

Following from this definition of the transform is the time-derivative rule

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-) \tag{3}$$

whereby the initial conditions existing before any $t = 0$ transient are brought into the analysis. Note that the 0^- indicates that the response will be calculated in terms of what we term the *pre-initial* conditions.

Also associated with this definition is the the initial-value theorem

$$\lim_{s\to\infty\cdot 1} sF(s) = f(0^+). \tag{4}$$

where the notation $\infty \cdot 1$ indicates that the limit is taken along the positive real axis. It is interesting to note here that the value calculated by (4) is associated with the *post-initial* values at $t = 0^+$. This form of the initial-value theorem is the correct result, and it is also the desired one, since we are mostly interested in the initial value *after* any discontinuities at $t = 0$.

More generally, if $F(s)$ is able to be written as a polynomial plus a function $\tilde{F}(s)$ converging to zero as $s \to \infty \cdot 1$ then

$$\lim_{s\to\infty\cdot 1} s\tilde{F}(s) = f(0+)\,. \tag{5}$$

These properties (3), (4), (5) and some extensions are developed more fully in the appendices.

With this overview in hand, the remainder of the paper is organized as follows: We first motivate the discussion with two simple dynamic system examples such as might be presented in a sophomore engineering course. The responses of these systems are calculated via the Laplace transform definition and properties presented above. Next, we discuss the application of the Laplace transform to abstract signals, independent of any dynamic systems context, to clarify the need for the consistent definitions presented above. Finally, the article concludes with appendices which introduce and develop the transform

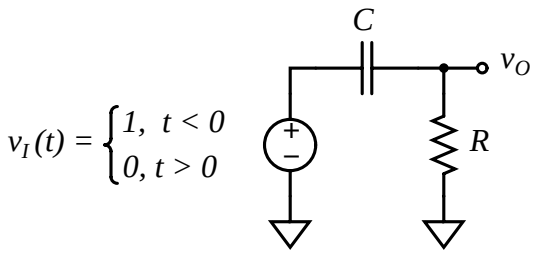

Fig. 1. Schematic of a high-pass electrical filter driven by an "unstep." The initial state of this system is the capacitor voltage $v_C(0^-) = 1$, and thus the initial output voltage is $v_O(0^-) = 0$.

properties with reasonable mathematical support. In order to do this, we need to think carefully about how singularity functions are defined, how these are combined with regular functions to form generalized functions, and how required mathematical operations on these generalized functions can be consistently defined.

I. Application Examples

In this section we present a pair of example problems drawn from electrical and mechanical engineering, respectively. To correctly calculate the transient response of these systems requires care in applying the transform (2); the set of properties presented above (3), (4), and (5) yield the correct answer.

A. First-Order High-Pass Filter Driven by an "Unstep"

First, consider the high-pass filter shown in Figure 1, which is driven by an "unstep" function

$$v_I(t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}.$$

The Laplace transform of this input is

$$\mathcal{L}\{v_I(t)\} = V_i(s) = 0,$$

which certainly seems uninteresting. We also specify the initial condition $v_O(0^-) = 0$, and thus the capacitor is initially charged to $v_C(0^-) = 1$V.

To find the total system response to the initial state and this input, we start with the differential equation

$$C\frac{d}{dt}(v_I - v_O) - \frac{v_O}{R} = 0$$

or

$$\frac{dv_O}{dt} + \frac{v_O}{RC} = \frac{dv_I}{dt}.$$

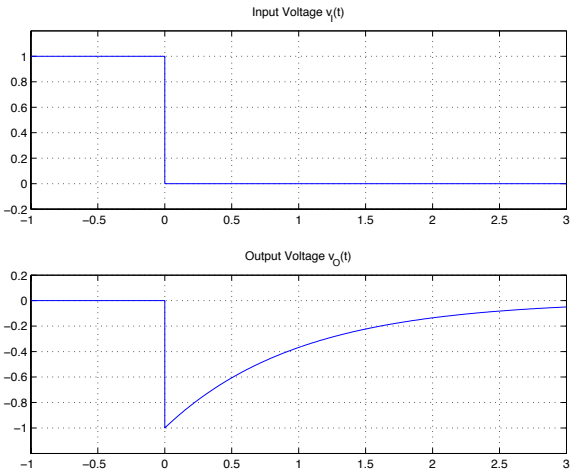

Fig. 2. Response of the high-pass filter driven by an "unstep."

The Laplace transform of this differential equation, using the correct form of the derivative rule, is

$$sV_o(s) - v_O(0^-) + \frac{V_o(s)}{RC} = sV_i(s) - v_I(0^-),$$

which reduces to

$$V_o(s) = \frac{sV_i(s) - v_I(0^-) + v_O(0^-)}{s + 1/RC}.$$

The associated pre-initial values are $v_I(0^-) = 1$ and $v_O(0^-) = 0$, and we have already calculated $V_i(s) = 0$. Thus the expression simplifies to

$$V_o(s) = \frac{-1}{s + 1/RC}. \tag{6}$$

Inverting this transform gives the output time waveform as

$$v_O(t) = -e^{-t/RC}; \quad t > 0,$$

which is shown in Figure 2 along with the input waveform. We can further apply the initial value theorem to (6) as:

$$v_O(0^+) = \lim_{s\to\infty} sV_o(s) = -1.$$

Of course, simple time-domain arguments lead to these same results with less effort, but our intention here is to demonstrate the proper application of Laplace techniques in solving such problems.

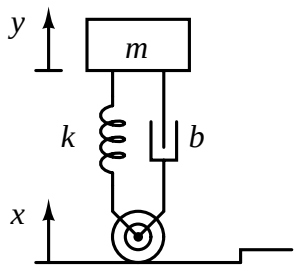

Fig. 3. An idealized automobile-suspension system approaching a rectangular speed bump. In the analysis, the center of the wheel is assumed to follow the bump exactly.

B. Second-Order Car Suspension

Another example which illuminates the proper use of the Laplace transform technique is the idealized second-order automobile-suspension system shown in Figure 3. In this simplified model, we assume that the center of the wheel follows the step contour exactly, such that the input motion $x(t)$ takes the form of a unit step. We are then interested in calculating the resulting car body output motion $y(t)$. We examine this response under three sets of initial conditions in order to give some insight into the types of solutions that might be calculated with the Laplace technique.

The differential equation describing the system is

$$m\ddot{y} = b(\dot{x} - \dot{y}) + k(x - y).$$

The Laplace transforms of the derivatives are

$$\mathcal{L}\{\dot{x}\} = sX(s) - x(0^-),$$

$$\mathcal{L}\{\dot{y}\} = sY(s) - y(0^-),$$

and

$$\mathcal{L}\{\ddot{y}\} = s^2Y(s) - sy(0^-) - \dot{y}(0^-).$$

Therefore, the differential equation transforms to

$$\begin{aligned} ms^2Y(s) - msy(0^-) - m\dot{y}(0^-) = \\ b\left(sX(s) - x(0^-) - sY(s) + y(0^-)\right) + \\ k\left(X(s) - Y(s)\right) \end{aligned}$$

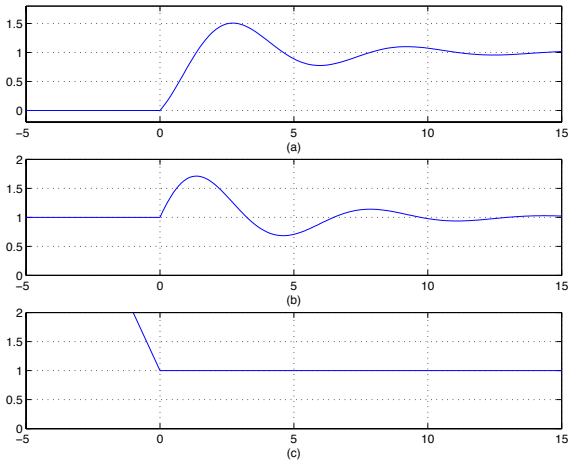

Fig. 4. Response of the idealized suspension system to an input unit step with three sets of initial conditions: (a) initial rest; (b) initial position $y(0^-) = 1$, zero initial velocity $\dot{y}(0^-) = 0$; and (c) initial position $y(0^-) = 1$ and initial velocity $\dot{y}(0^-) = -b/m$. The last set of initial conditions results in immediate convergence to final value.

Solving for the output $Y(s)$ gives

$$Y(s) = \frac{(bs+k)}{ms^2+bs+k}X(s) + \frac{-bx(0^-) + (ms+b)y(0^-) + m\dot{y}(0^-)}{ms^2+bs+k}$$

Using this solution, we can properly find the system response to arbitrary inputs and initial conditions.

We now assume that the input takes the form of a unit step $x(t) = u(t)$, which has a pre-initial value $x(0^-) = 0$. The transform is $X(s) = 1/s$. The solution under this unit step is then calculated for three sets of initial conditions. The first set of initial conditions (a) are zero (initial rest): $y(0^-) = 0$ and $\dot{y}(0^-) = 0$. The resulting output transform is

$$Y(s) = \frac{bs+k}{s(ms^2+bs+k)}.$$

Inverting this result yields the zero-state step response shown in Figure 4(a). Note that in this system, for a step input, the position is continuous across $t = 0$, and the velocity takes a step with value b/m m/sec. This is true independent of initial conditions and thus applies to the three cases studied here.

The second set of initial conditions (b) are initial position $y(0^-) = 1$, zero initial velocity $\dot{y}(0^-) = 0$. That is, the system position starts at the level which it will settle to in steady-state. In this case, the output transform expression is

$$Y(s) = \frac{ms^2+2bs+k}{s(ms^2+bs+k)},$$

or

$$Y(s) = \frac{1}{s} + \frac{b}{ms^2+bs+k}.$$

Inverting this expression gives the time response shown in Figure 4(b). The observed motion results from the force impulse $b\delta(t)$ which the damper applies to the mass at the moment of the step. Applying the initial value theorem to $Y(s)$ and $sY(s)$, respectively yields the post-initial values $y(0+) = 1$, and $\dot{y}(0^+) = b/m$. The post-initial velocity is simply an expression of the change in momentum induced in the mass by the force impulse applied from the damper; this change in momentum occurs in all three cases studied here.

The third set of initial conditions is selected to yield immediate convergence to the final value of the response. That is, we choose the initial position as $y(0^-) = 1$, and set the initial velocity to $\dot{y}(0^-) = -b/m$, such that the initial velocity will be exactly canceled by the force impulse from the damper. For these initial conditions, the output transform is

$$Y(s) = \frac{ms^2 + bs + k}{s(ms^2 + bs + k)} = \frac{1}{s}.$$

We recognize this as the transform of the unit step, which gives

$$y(t) = 1; \quad t > 0.$$

This waveform is shown in Figure 4(c), along with a pre-initial velocity of $-b/m$, shown by the straight line segment of position for $t < 0$. Of course, in the analysis we do not concern ourselves with exactly how this pre-initial velocity is established, nor any details of the time variation prior to the transient. The only required prior knowledge is the velocity and position immediately before the transient ($t = 0^-$).

II. Signal Examples

While the Laplace transform is frequently associated with the solution of differential equations, the need to clearly distinguish 0^+ and 0^- is independent of any dynamic systems context. In the following example, adapted from Problem 11.17 in Siebert [4], we apply the unilateral transform to three signals and their derivatives. This clarifies that the need for using the $\mathcal{L}_-$ form (2), (3), (4), and (5) is really a matter of properly defining signals and their transforms, and is not fundamentally connected to the solution of differential equations.

Consider the three signals $f(t)$, $g(t)$, and $h(t)$ as shown in Figure 5,

$$\begin{aligned} f(t) &= e^{-at} \\ g(t) &= e^{-at}u(t) \\ h(t) &= e^{-at}u(t) - u(-t), \end{aligned}$$

which are plotted for the value $a = 1$. All three functions are nonsingular and agree for positive time, therefore they all have the same Laplace transform. However, their derivatives differ for $t \leq 0$. In particular, the derivatives include differing amounts of an impulse at $t = 0$, and thus the Laplace transforms of their derivatives must differ. Our choice of Laplace transform properties should give consistent and correct results when operating on these signals and their derivatives. The associated transforms are calculated below to show that this is the case. We also demonstrate the consistent use of the initial value theorem in the context of these signals.

A. Function $f(t) = e^{-at}$

Consider the function $f(t) = e^{-at}$ with associated pre-initial value $f(0^-) = 1$. The Laplace transform of $f(t)$ is

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}.$$

The time derivative of $f(t)$ is

$$f'(t) = -ae^{-at},$$

and the Laplace transform of the time derivative is

$$\mathcal{L}\{-ae^{-at}\} = \frac{-a}{s+a}. \tag{7}$$

Using the derivative rule

$$sF(s) - f(0^-) = \frac{s}{s+a} - 1 = \frac{-a}{s+a}$$

produces the same result. The results from the initial-value theorem are

$$f(0^+) = \lim_{s \to \infty \cdot 1} \frac{s}{s+a} = 1,$$

and

$$f'(0^+) = \lim_{s \to \infty \cdot 1} \frac{-sa}{s+a} = -a.$$

There is consistency between the time domain and Laplace domain calculation of the signals and their initial value.

B. Function $g(t) = e^{-at}u(t)$

The function $g(t) = e^{-at}u(t)$ has an associated pre-initial value $g(0^-) = 0$. The Laplace transform of $g(t)$ is the same as for $f(t)$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}.$$

However, the time derivative now includes an impulse

$$g'(t) = \delta(t) - ae^{-at}u(t).$$

The Laplace transform of this time derivative is

$$\mathcal{L}\{g'(t)\} = 1 - \frac{a}{s+a} = \frac{s}{s+a}$$

which is different from the result above (7). Using the correct derivative rule

$$sG(s) - g(0^-) = \frac{s}{s+a} - 0 = \frac{s}{s+a}$$

we get a consistent result. The initial-value theorem gives

$$g(0^+) = \lim_{s\to\infty\cdot 1} \frac{s}{s+a} = 1$$

producing the correct value at $t = 0^+$. We can also apply the more general initial value theorem (5) to the transform of the derivative as follows (see the discussion in the appendix for more details). Expanding out the nonsingular part of the transform gives

$$G(s) = 1 - \frac{a}{s+a} \equiv 1 + \tilde{F}(s).$$

And thus,

$$g'(0^+) = \lim_{s\to\infty\cdot 1} sF(s) = -a$$

which is the correct value.

C. Function $h(t) = e^{-at}u(t) - u(-t)$

Finally consider the function

$$h(t) = \begin{cases} -1 & t < 0 \\ e^{-at} & t > 0 \end{cases} = e^{-at}u(t) - u(-t),$$

which has an associated pre-initial value $h(0^-) = -1$. The Laplace transform of this signal is the same as for the other two ($H(s) = G(s) = F(s)$), and so we don't list it here. Computing the time derivative gives

$$h'(t) = 2\delta(t) - ae^{-at}u(t).$$

The Laplace transform of this time derivative is

$$\mathcal{L}\{h'(t)\} = 2 - \frac{a}{s+a} = \frac{2s+a}{s+a}.$$

Using the correct derivative rule

$$sH(s) - h(0^-) = \frac{s}{s+a} + 1 = \frac{2s+a}{s+a}$$

gives a consistent result. Finally, the initial value theorem gives a correct result for both h and its derivative, $h(0^+) = 1$ and $h'(0^+) = -a$, although we don't show the details here.

In conclusion, the formulas (2), (3), (4), and (5) give correct results. We hope that the signal examples presented above help to clarify the application of these formulas.

III. The Horror of 0^+

Several versions of the Laplace transform are often found in the literature, which differ in the lower limit of integration. Some books quote the Laplace integral as

$$\mathcal{L}_0\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{8}$$

which creates an ambiguity with respect to signularities at the origin.

Other books solve the ambiguity at the origin by specifying the 0^+ version of the transform

$$\mathcal{L}_+\{f(t)\} = \int_{0^+}^\infty f(t)e^{-st}\,dt. \tag{9}$$

To differentiate between this form and the prefered form

$$\mathcal{L}_-\{f(t)\} = \int_{0^-}^\infty f(t)e^{-st}\,dt$$

these forms of the transform are often referred to as $\mathcal{L}_-$ and $\mathcal{L}_+$ [5].

Using the $\mathcal{L}_+$ form of the Laplace transform renders the waveforms in Section II and their derivatives indistinguishable. A consistent set of Laplace-transform properties can be constructed using $\mathcal{L}_+$, namely,

$$\mathcal{L}_+\{f(t)\} = \int_{0^+}^\infty f(t)e^{-st}\,dt$$

$$\mathcal{L}_+\{f'(t)\} = sF(s) - f(0^+)$$

and

$$\lim_{s\to\infty} sF(s) = f(0^+).$$

Note that 0 appears as 0^+ in all of the above equations. There is a symmetric, perhaps attractive, consistency in this appearance. However, this form of the Laplace transform produces unuseful, trivial answers to many engineering problems. Most embarrassingly, note that

$$\mathcal{L}_+\{\delta(t)\} = \int_{0^+}^{\infty} \delta(t)e^{-st}\,dt = 0$$

which should strike the reader as utter nonsense, at least for practical applications.

IV. Conclusion

For a consistent set of properties for the unilateral Laplace transform that are useful to engineers, use the definition

$$\mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt,$$

the derivative rule

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-),$$

and the initial-value theorem

$$\lim_{s\to\infty} sF(s) = f(0^+).$$

These properties give students general tools that can be used to solve differential equations with both non-zero initial conditions and inputs. These also work properly in the presence of input singularities. Other presentations of the unilateral transform are not acceptable.

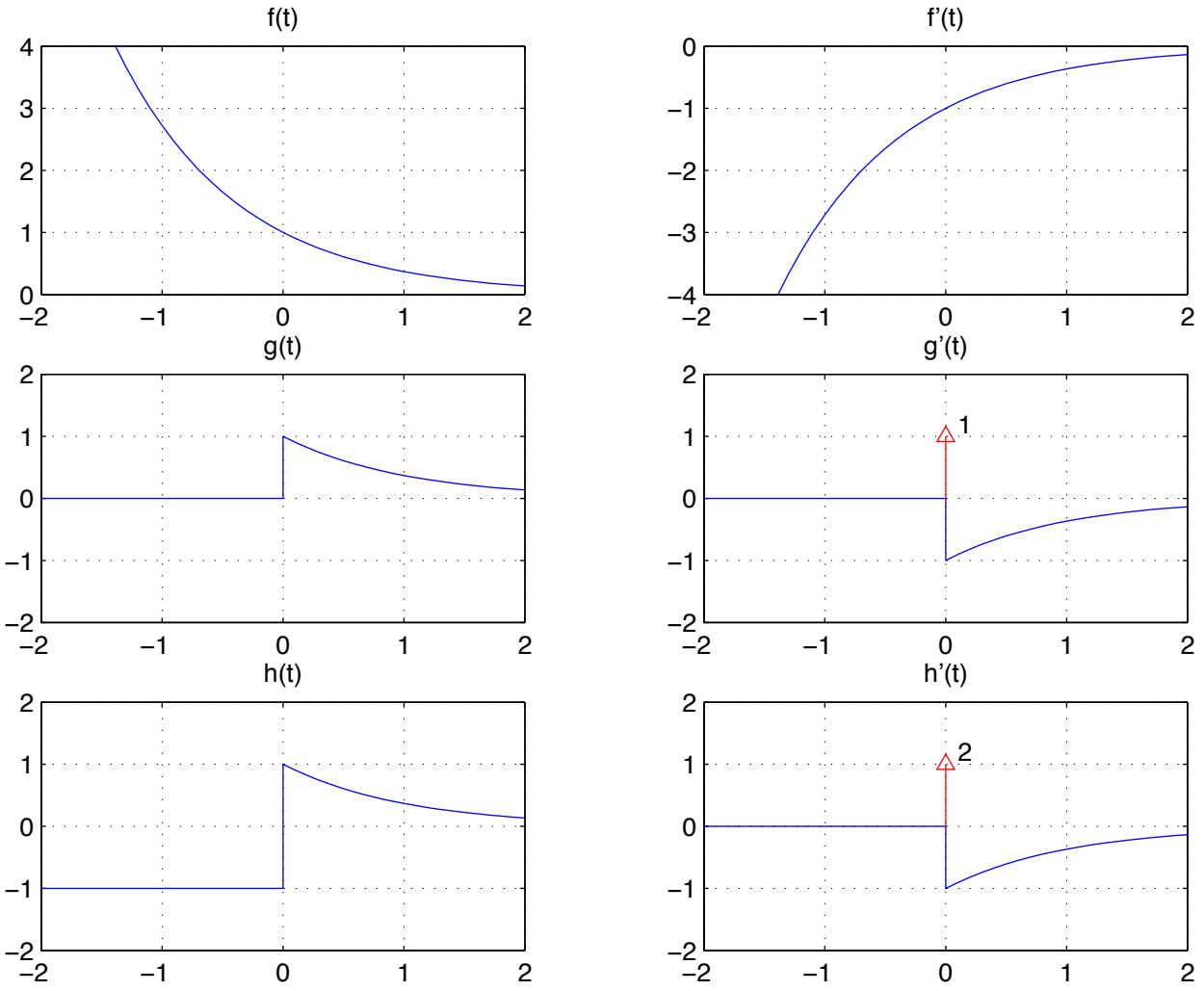

Fig. 5. Three functions $f(t) = e^{-at}$, $g(t) = e^{-at}u(t)$, $h(t) = e^{-at}u(t) - u(-t)$, plotted for $a = 1$, and their derivatives, defined for all time. These waveforms are used for illuminating transform properties.

APPENDICES

Many treatments of the unilateral Laplace transform in the textbook literature are self-contradictory. These inconsistencies arise from a reluctance to use the generalized derivative; a consistent and easily understood treatment can be based on a uniform use of the generalized derivative. In the following appendices, we provide a reasonable mathematical foundation for the use of generalized functions in association with dynamic systems and, in particular, in the context of the Laplace transform. The book by Hoskins [1] gives further perspective on the issues presented below.

APPENDIX I
GENERALIZED FUNCTIONS

Discontinuous signals are a part of life. Generalized functions arise in any consistent treatment of their derivatives. One requirement is a uniform description of the Laplace transform of the derivative.

In many textbooks dealing with the one-sided Laplace transform, "rest initial conditions" are taken as a standard case. At the least these conditions include $f(0^-) = 0$, so any signal $f(t)$ with $f(0^+) \neq 0$ presents us with a discontinuity at $t = 0$. This assumption is overly restrictive, as we should expect a consistent set of Laplace definitions to admit non-zero pre-initial conditions on states and signals. The rest initial condition is just a special case.

Most elementary textbooks, both in mathematics and in engineering, attempt to minimize the discussion of singularity functions by giving any $t = 0$ discontinuity special treatment. Along the way, they disguise the fact that there is a discontinuity at zero at all, and this exacerbates the confusion of functions such as 1 and the unit step function (which have the same Laplace transform and the same value $f(0^+)$ but which have derivatives with different Laplace transforms).

The treatment of generalized functions presented below is just a small part of the standard mathematical theory of distributions, but what we present is both sufficiently concrete to be quickly understood, and sufficiently general to be widely useful. We will describe the *behavior* of singularity functions in a way which we hope supports the common intuition about them. Because of our focus on the one-sided Laplace transform, we will only consider generalized functions on the interval $[0, \infty)$, and we incorporate into the information carried by a generalized function the values at 0^- of the function and all its derivatives.

APPENDIX II

DEFINITION OF GENERALIZED FUNCTIONS

A function $f(t)$ on $[0,\infty)$ is *piecewise continuous* if there is a sparse set $S \subset [0,\infty)$ such that $f(t)$ is continuous on the complement of S, and for every $a \in [0,\infty)$ the limits $f(a+)$ and $f(a-)$ exist. (A set S of numbers is *sparse* if no interval of finite width contains infinitely many of its elements.)

This last clause requires special interpretation in case $a = 0$. In this case we are still demanding the value of $f(0-)$, but this is now additional information, not determined by the values of the function on $[0,\infty)$. This might be called a "pre-initial condition," in contrast with $f(0+)$ which is "post-initial."

A function $f(t)$ on an interval $[0,\infty)$ is *piecewise smooth* if there is a sparse set $S \subset [0,\infty)$ such that all derivatives of $f(t)$ exist on the complement of S, and are themselves piecewise continuous functions on $[0,\infty)$.

Thus part of the data of a piecewise smooth function on $[0,\infty)$ is the sequence of values $f^{(n)}(0-)$, $n = 0, 1, 2 \ldots$. This data is not implied by values of $f(t)$ for $t \geq 0$.

In this setting, the Heaviside unit step function $u(t)$ is defined to take on the value 1 for all $t > 0$ and have zero pre-initial conditions, $u^{(n)}(0-) = 0$ for all $n \geq 0$. We leave the reader to choose a value for $u(0)$; the value of a generalized function at any single point is not significant, as explained below.

A *singularity function* is a formal sum of the form

$$f_s(t) = \sum_{k,l} c_{k,l}\delta^{(l)}(t - a_k)$$

where $\delta^{(l)}$ is the l^{th} derivative of the unit impulse. Here a_k is an increasing sequence in $[0,\infty)$. For each value of k, only finitely many of the $c_{k,l}$ are allowed to be nonzero. The sum can be over a finite or infinite range of values of k. If it is infinite, we require that $\lim_{k\to\infty} a_k = \infty$.

The a_k for which some $c_{k,l}$ is nonzero are the "singular points" of $f_s(t)$. The "singularity at $t = a$" of $f_s(t)$ is the part of this sum having $a_k = a$. If this sum is empty, $f_s(t)$ is said to be "nonsingular at $t = a$."

We also comment here that the unit step function $u(t) = \delta^{(-1)}(t)$ is taken as unitless. Succeeding derivatives introduce increasing negative powers of time. The units of the singularity functions $\delta^{(l)}(t)$ are thus $\sec^{-(l+1)}$. If we wish to use these functions in association with physical quantities, then the functions must be multiplied by weighting terms with appropriate units. For example, in a voltage waveform, the functions $\delta^{(l)}(t)$ must be multiplied by weighting terms $c_{k,l}$ which have units of Volt-$\sec^{(l+1)}$. As another example, a force impulse $F_0\delta(t)$ has a weighting term F_0 with units of N-sec. The term F_0 then

corresponds to the area under a finite-duration force event, such as a hammer blow, which we might model with an impulse function.

A *generalized function* is a formal expression

$$f(t) = f_r(t) + f_s(t)$$

whose "regular part" $f_r(t)$ is determined by a piecewise smooth function and whose "singular part" $f_s(t)$ is a singularity function. The regular part contributes the data $f^{(n)}(0-) = f_r^{(n)}(0-)$.

There is a sublety here. Two piecewise smooth functions determine the same generalized functions if (and only if) they differ from each other on a sparse set. This is why the value of the Heaviside function at $t = 0$ is irrelevant. This caveat is necessary in order for the usual uniqueness theorem for solutions of differential equations to hold for generalized functions.

Generalized functions can be differentiated to give other generalized functions. The "generalized derivative" of a piecewise smooth function $f_r(t)$ is given by the ordinary derivative of $f_r(t)$ at its smooth points, along with the specification that $(f')^{(n)}(0-) = f_r^{(n+1)}(0-)$, plus the sum of delta functions of the form

$$(f(a+) - f(a-))\delta(t-a)$$

for each point a at which $f(t)$ is discontinuous. The derivative of a singularity function simply follows the notation $\delta'^{(l)} = \delta^{(l+1)}$.

Generalized functions cannot generally be multiplied together, and such general products never occur in applications of the Laplace transform. However, if $f(t)$ is a smooth function then its product with any generalized function $g(t)$ is defined, and the "product rule"

$$(f \cdot g)'(t) = f'(t)g(t) + f(t)g'(t)$$

holds in this generality. Indeed, this rule is used to derive the definition of the product. We start with the relation

$$f(t)\delta(t-a) = f(a)\delta(t-a). \tag{10}$$

We can use this and the product rule to express all other products of the form $f(t)\delta^{(n)}(t-a)$, where $f(t)$ is a smooth function, as linear combinations of $\delta^{(n-k)}(t-a)$'s with constant coefficents. To see this, differentiate (10) and apply this same identity with $f'(t)$ replacing $f(t)$ to find

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(a)\delta(t-a).$$

Continuing in the same way leads by induction to the identity

$$f(t)\delta^{(n)}(t-a) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} f^{(k)}(a)\,\delta^{(n-k)}(t-a).$$

This formula specifies how a generalized function is to be multiplied by a smooth function $f(t)$. It also explains exactly how smooth $f(t)$ has to be before the product $f(t)\delta^{(n)}(t-a)$ makes sense: it and its derivatives up to $f^{(n)}(t)$ should exist at $t=a$ and be continuous there. This result also appears in Hoskins [1] as equation 3.26.

The product rule can be integrated to get the usual integration by parts formula, valid for any smooth function $f(t)$ and any generalized function $g(t)$:

$$\int_{a-}^{b+} f(t)g'(t)\,dt = f(t)g(t)\Big|_{a-}^{b+} - \int_{a-}^{b+} f'(t)g(t)\,dt$$

If we apply this to the case $a=0$, we will use the values $f(0-)$ and $g(0-)$ which come as part of the definition of these functions. Part of the definition of being smooth (as opposed to piecewise smooth) is that $f(0-)=f(0)$, so:

$$\int_{0-}^{b+} f(t)g'(t)\,dt = (f(b)g(b+) - f(0)g(0-)) - \int_{0-}^{b+} f'(t)g(t)\,dt.$$

APPENDIX III

EQUATION (3), THE DERIVATIVE RULE

An important feature of generalized functions is that (assuming appropriate growth conditions) one can define their Laplace transforms. The basic singularity functions satisfy

$$\mathcal{L}\{\delta^{(n)}(t)\} = s^n$$

It's easy to check that the differentiation rule is compatible with this.

The derivative rule (3)

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-).$$

can be directly derived from the definition (2) through integration by parts:

$$\mathcal{L}\{f'(t)\} = \int_{0^-}^{\infty} e^{-st} f'(t)\,dt\,.$$

Integrating by parts

$$\int u\,dv = uv - \int v\,du$$

$$u = e^{-st} \qquad dv = f'(t)\,dt$$

$$du = -se^{-st}\,dt \qquad v = f(t)$$

results in

$$\begin{aligned}\int_{0^-}^{\infty} & e^{-st} f'(t)\, dt \\ &= e^{-st} f(t)\Big]_{0^-}^{\infty} + s\int_{0^-}^{\infty} e^{-st} f(t)\, dt \\ &= f(0^-) + s\mathcal{L}\{f(t)\}\end{aligned}$$

Provided one systematically uses the generalized derivative, this formula is valid for any generalized function $f(t)$ provided s has real part large enough for the improper integrals to converge.

APPENDIX IV
EQUATION (4), THE INITIAL-VALUE THEOREM

The initial-value theorem is

$$\lim_{s\to\infty\cdot 1} sF(s) = f(0^+).$$

Actually, it is more accurate to call this the post-initial-value theorem, since it yields the result at $0+$, but we will stay with standard terminology and refer to it as the initial value theorem. This result can be derived via several approaches which provide alternate insights. We show three approaches below, which are based upon a) a formal application of the derivative rule, b) a qualitative argument from the sifting property of se^{-st} in the limit, and c) the *initial singularity theorem.*

A. From the Derivative Rule

The initial-value theorem (4) can be derived from the derivative rule (3). If

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0^-)$$

then

$$sF(s) = \int_{0^-}^{\infty} f'(t)e^{-st}\, dt + f(0^-).$$

Taking the limit as s goes to infinity along the real axis gives

$$\begin{aligned}\lim_{s\to\infty\cdot 1} sF(s) &= \lim_{s\to\infty\cdot 1}\left(\int_{0^-}^{\infty} f'(t)e^{-st}\, dt\right) + f(0^-) \\ &= \lim_{s\to\infty\cdot 1}\left(\int_{0^-}^{0^+} f'(t)e^{0}\, dt + \int_{0^+}^{\infty} f'(t)e^{-st}\, dt\right) + f(0^-) \\ &= \lim_{s\to\infty\cdot 1}\left(f(t)\Big]_{0^-}^{0^+} + 0\right) + f(0^-) = f(0^+)\end{aligned}$$

which is the expected result.

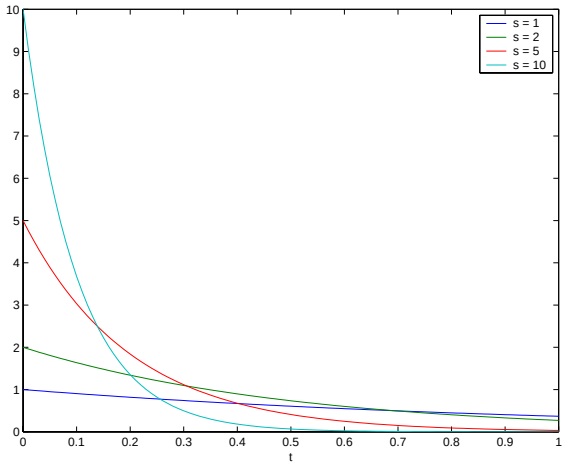

Fig. 6. Sifting property of se^{-st}: As s grows toward large positive real values, se^{-st} approaches a delta function on the positive side of the origin.

B. From the Sifting Property

The initial-value theorem can also be heuristically argued, as suggested by Kailath [8], by multiplying the definition by s

$$sF(s) = \int_{0^-}^{\infty} f(t)\, se^{-st}\, dt$$

and taking the limit along the real axis

$$\lim_{s \to \infty \cdot 1} sF(s) = \lim_{s \to \infty \cdot 1} \int_{0^-}^{\infty} f(t)\, se^{-st}\, dt.$$

As suggested by the plot in Figure 6, as s grows larger, the function se^{-st} sifts for values $f(0^+)$, which we indicate via the notation

$$\lim_{s \to \infty \cdot 1} se^{-st} \longrightarrow \delta(t - 0^+).$$

Therefore

$$\lim_{s \to \infty \cdot 1} sF(s) = \int_{0^-}^{\infty} f(t)\, \delta(t - 0^+)\, dt = f(0^+)$$

as expected.

C. From the Initial Singularity Formula

The initial-value theorem can perhaps best be thought of as a special case of what we term the *Initial Singularity Theorem*, which asserts that $F(s)$ is asymptotic, as s increases through real numbers, to a

polynomial which carries precisely the information about the singularity of $f(t)$ at $t = 0$. To express this we will use the notation $F(s) \sim G(s)$ to mean

$$\lim_{s \to \infty \cdot 1} (F(s) - G(s)) = 0 \,.$$

The initial singularity formula asserts that if

$$\sum_l c_l \delta^{(l)}(t)$$

is the singularity of $f(t)$ at $t = 0$, and $F(s) = \mathcal{L}\{f(t)\}$, then

$$F(s) \sim \sum_l c_l s^l \,.$$

This comes out of the value $\mathcal{L}\{\delta^{(n)}(t)\} = s^n$ together with the two facts

$$\lim_{s \to \infty \cdot 1} \mathcal{L}\{f_r(t)\} = 0 \,,$$

and

$$\lim_{s \to \infty \cdot 1} \mathcal{L}\{\delta^{(n)}(t-a)\} = 0 \quad \text{for} \quad a > 0 \,.$$

The Initial Value Theorem arises by applying the Initial Singularity Theorem to $f'(t)$. The singularity of $f'(t)$ at $t = 0$ is

$$(f(0+) - f(0-))\delta(t) + \sum_l c_l \delta^{(l+1)}(t) \,,$$

with Laplace transform

$$(f(0+) - f(0-)) + \sum_l c_l s^{l+1} \,,$$

so the t-derivative rule (3) implies

$$sF(s) - f(0-) = \mathcal{L}\{f'(t)\} \sim (f(0+) - f(0-)) + \sum_l c_l s^{l+1} \,.$$

Canceling the $f(0-)$'s,

$$sF(s) \sim f(0+) + \sum_l c_l s^{l+1} \,.$$

In particular, if $f(t)$ is nonsingular at $t = 0$, then

$$\lim_{s \to \infty \cdot 1} sF(s) = f(0+) \,.$$

More generally, with the above analysis, we can see that the value $f(0+)$ exists even if there is a singularity at $t = 0$, as also noted by Kailath [8]. That is, if $F(s)$ is a polynomial plus a function $\tilde{F}(s)$ converging to zero as $s \to \infty \cdot 1$ then

$$\lim_{s \to \infty \cdot 1} s\tilde{F}(s) = f(0+) \,.$$

For example, consider the first-order transform

$$F(s) = \frac{s+a}{s+b} = 1 + \frac{a-b}{s+b}.$$

This has a post-initial value $f(0^+) = a - b$, even though $f(t)$ includes an impulse at $t = 0$, as can be seen using our result above, or via direct inversion of the transform.

APPENDIX V
INVERSE

Many books quote the inverse Laplace transform

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}F(s)\,ds.$$

However, this result is primarily of theoretical interest, and we have not found use for it in undergraduate teaching. The limited-order systems typically studied with Laplace techniques are readily solved via partial fraction expansion and inversion via tables. As well, this formula requires the use of complex integration, which is not a required component of most undergraduate engineering curricula. Thus we do not recommend more than a brief mention of this result in undergraduate engineering courses.

APPENDIX VI
CONFUSION IN THE TEXTBOOK LITERATURE

The textbook literature is surprisingly random in its treatment of the unilateral Laplace transform; many otherwise-excellent texts fumble this issue. As is clear from the above discussions, we regard the $\mathcal{L}_-$ from as the only suitable approach for the study of dynamic systems. This form is properly adopted in a number of texts, for example, those of Franklin, Powell, and Emami-Naeini [12], DeCarlo and Lin [13], Kailath [8], and Siebert [4]. We have found the discussion in Siebert [4], Chapters 2 and 11, particularly helpful. Our example waveforms in section II are based on Problem 11.17 in Siebert's text.

Many authors use the $\mathcal{L}_+$ version of the unilateral Laplace transform. This unfortunate choice can likely be traced back to the classic 1942 text by Gardner and Barnes [9]. The title of this paper,"Troubles at the Origin," was inspired by section 3-6 of Aseltine [10]. However, while this text correctly identifies the problem, it reaches the wrong conclusion! The $\mathcal{L}_+$ version of the transform also seems preferred in mathematically-oriented treatments such as the frequently-cited book by Zemanian [2]. The control systems book by D'Azzo and Houpis adopts the $\mathcal{L}_+$ transform in the definition of the derivative property, but is a bit unclear as to the lower limit on the Laplace integral.

However, the $\mathcal{L}_+$ form of the transform has the huge disadvantage that it leaves transients at $t = 0$ as essentially an exercise for the reader, presumably to be solved by some time-domain technique such as impulse-matching. The example problems in section I could not be efficiently solved with the $\mathcal{L}_+$ form; with this form the transform of the unit impulse is identically zero! This little "problem" is frequently circumvented by defining the delta function as occurring completely to the right of $t = 0$. What then does $0+$ mean to these authors, and what time value is associated with the initial value theorem? We surely can't teach our engineering students to study transients via the $\mathcal{L}_+$ transform. We also owe it to them not to leave this issue fuzzy by avoiding discussion of 0^+ and 0^-.

Some books add to the confusion by showing both $\mathcal{L}_+$ and $\mathcal{L}_-$ forms, with little or no comment as to when to chose between them. This includes the texts by Ogata [5] [6]. The book by Close and Frederick [3] states that both forms are acceptable as long as the corresponding properties are developed in a consistent fashion. However, this leads to some convoluted reasoning associated with the transform of the unit impulse and with the derivative property.

Furthermore, textbooks that simply claim $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ or $\lim_{s\to\infty} sF(s) = f(0)$ without further clarification must be considered suspect (even otherwise fine math books such as Churchill [11]). The systems book by Palm [7] states that using the one-sided transform is the same as assuming that signals are zero for $t < 0$, which is not true.

In summary, while the books referenced above have much to recommend them, the issue of Laplace techniques requires clarification in many of these references, and in our teaching of systems. We hope that the discussion in this paper will be helpful toward that end.

REFERENCES

[1] Roland F. Hoskins. *Delta Functions: An Introduction to Generalised Functions*. Horwood Publishing Limited, West Sussex, England, 1999.

[2] A. H. Zemanian. *Distribution Theory and Transform Analysis*. McGraw-Hill, New York, NY, 1965.

[3] Charles M. Close and Dean K. Frederick. *Modeling and Analysis of Dynamic Systems, 2^{nd} Ed.* John Wiley and Sons, New York, NY, 1995.

[4] William McC. Siebert. *Circuits, Signals, and Systems*, pages 357–358. MIT Press, Cambridge, MA, 1986.

[5] Katsuhiko Ogata. *Modern Control Engineering, 4^{th} Ed.* Prentice-Hall, Englewood Cliffs, NJ, 2002.

[6] Katsuhiko Ogata. *System Dynamics, 4^{th} Ed.*. Prentice-Hall, Englewood Cliffs, NJ, 2004.

[7] William J. Palm III. *Modeling, Analysis, and Control of Dynamic Systems, 2^{nd} Ed.*. John Wiley and Sons, New York, NY, 2000.

[8] Thomas Kailath. *Linear Systems*, page 12. Prentice-Hall, Englewood Cliffs, NJ, 1980.

[9] Murray F. Gardner and John L. Barnes. *Transients in Linear Systems*. Wiley, New York, NY, 1942.

[10] John A. Aseltine. *Transform Method in Linear Systems Analysis*, pages 27–29. McGraw-Hill, New York, NY, 1958.

[11] Ruel V. Churchill. *Complex Variables and Applications*. McGraw-Hill, New York, NY, 1960.

[12] Gene Franklin, David Powell, and Abbas Emami-Naeini. *Feedback Control of Dynamic Systems*, 4^{th} *Ed.* Prentice-Hall, Upper Saddle River, NJ, 2002.

[13] Raymond A. DeCarlo and Pen-Min Lin. *Linear Circuit Analysis*, 2^{nd} *Ed*. Oxford University Press, New York, NY, 2001.