2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 8

Kinematics of Rigid Bodies

Instant Centers or Instantaneous Centers

"Point on a rigid body whose velocity is zero at a given instant"
Instantaneous: May only have zero velocity at the instant under consideration.
Idea: If we know the location of an instant center in 2D motion and we know the angular velocity of the rigid body, the velocities of all other points are easy to determine.

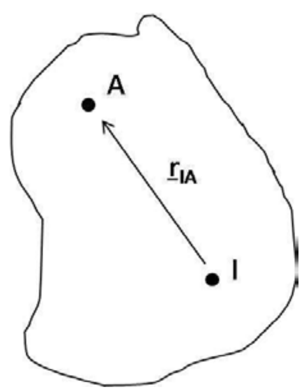

Figure 1: Rigid body where I is the instant center. Figure by MIT OCW.

$$\begin{aligned} \underline{v}_A &= v_I + \underline{\omega} \times \underline{r}_{IA} \\ &= 0 + \underline{\omega} \times \underline{r}_{IA} \end{aligned} \tag{1}$$

$$\underline{v}_A = \underline{\omega} \times \underline{r}_{IA}$$

So body is undergoing rigid body rotation about I.

How to locate an instant center?
1. Draw lines in direction of motion of two points on a rigid body.
2. Draw perpendiculars to both these lines at points.

3. I is the intersection of perpendiculars.

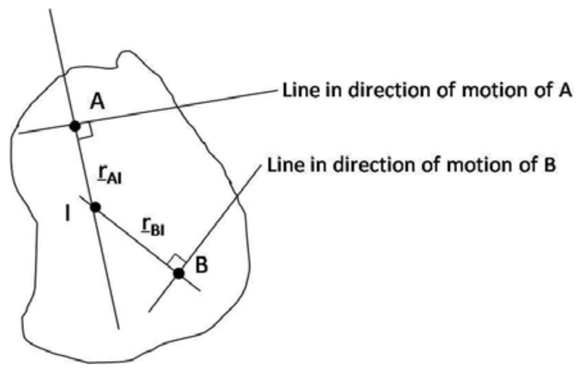

Figure 2: Rigid body with lines drawn to demonstrate how to find instant center. Figure by MIT OCW.

<u>Proof</u>

$$\underline{v}_I = \underline{v}_A + \underline{\omega} \times \underline{r}_{AI}$$

($\underline{\omega}$ is angular velocity of rigid body.)

$\underline{\omega} \times \underline{r}_{AI}$: Vector is $\perp$ to $\underline{r}_{AI}$. Direction of motion is parallel to motion of A.

$$\underline{v}_I = \underline{v}_B + \underline{\omega} \times \underline{r}_{BI}$$

$\underline{\omega} \times \underline{r}_{BI}$: Vector is $\perp$ to $\underline{r}_{BI}$. Direction of motion is parallel to motion of B.

Cannot satisfy both of these unless $\underline{v}_I = 0$. (I is the instant center.)

Note: The instant center may not actually be a point on the rigid body. This simply means that the body is instantaneously rotating about an external point. The instant center can change in time.

Example: Rolling Disk

What is the velocity of point A?

Assume no slip.

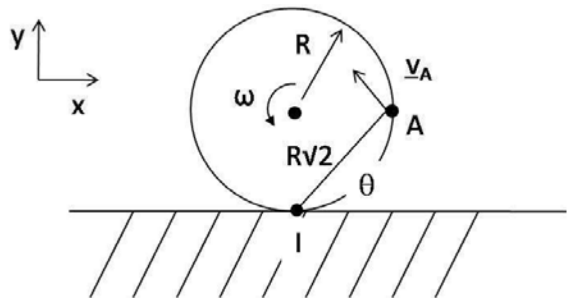

Figure 3: Rolling Disk. The disk rolls with angular velocity ω in the x-direction. Figure by MIT OCW.

Point I marks the contact with disk and ground: no slip, not moving for an instant because ground is still. → Instant center.

$$\begin{aligned} \underline{v}_A &= \underline{\omega}\hat{e}_z \times \underline{r}_{IA} \\ &= \underline{\omega}\hat{e}_z \times (-\sqrt{2}R\cos 45^\circ \hat{e}_x - \sqrt{2}R\sin 45^\circ \hat{e}_y) \end{aligned} \quad (2)$$

$$\begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & 0 & \omega \\ R\sqrt{2}\cos 45^\circ & R\sqrt{2}\sin 45^\circ & 0 \end{vmatrix} =$$

$$\hat{e}_x(-\sqrt{2}R\omega\sin 45^\circ) - \hat{e}_y(-\sqrt{2}R\omega\cos 45^\circ) = \boxed{-\sqrt{2}R\omega\sin 45^\circ \hat{e}_x + \sqrt{2}R\omega\cos 45^\circ \hat{e}_y = \underline{v}_A}$$

Example: Multibar Linkage

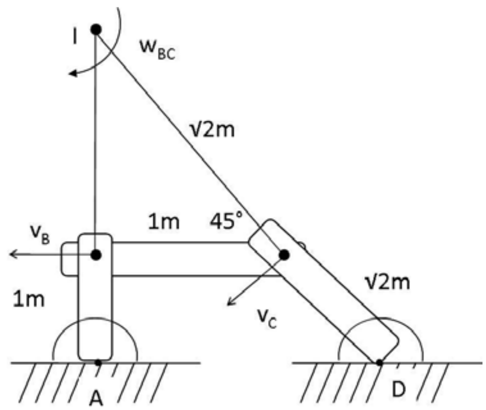

Figure 4: Multibar Linkage. Figure by MIT OCW.

Without Instant Centers

A is stationary.

$$\underline{v}_B = \underline{v}_A + \underline{\omega}_{AB} \times \underline{r}_{AB}$$
$$\underline{v}_C = \underline{v}_B + \underline{\omega}_{BC} \times \underline{r}_{BC}$$
$$\underline{v}_C = \underline{v}_D + \underline{\omega}_{CD} \times \underline{r}_{DC}$$

3 unknowns ($\underline{v}_C$, $\underline{\omega}_{BC}$, $\underline{\omega}_{CD}$), 3 equations. Take cross products and solve.

With Instant Centers

We know the directions of motion of B and C

$$\underline{v}_B = 10\frac{m}{s}(\text{From motion of bar AB}) = \omega_{BC} \cdot 1m$$

Therefore:

$$\boxed{\underline{\omega}_{BC} = 10\frac{rad}{s}(-\hat{e}_z)}$$

Now we know that:

$$\underline{v}_C = \omega_{BC} \cdot m\sqrt{2} = \sqrt{2} \cdot 10m/s$$

$$\underline{v}_C = \sqrt{2} \cdot 10 = \omega_{CD}\sqrt{2}$$

$$\boxed{\underline{\omega}_{CD} = 10 rad/s (+\hat{e}_z)}$$

Of course, this is a nice geometry → can easily become more complicated.

⇒ Good method for multibar linkages, and ladders against walls.

Plane Kinetics (Dynamics) Of Rigid Bodies

Recall that for a system of particles

1. Linear Momentum: $\frac{d}{dt}\underline{P} = \underline{F}_{ext}$
2. Angular Momentum: $\frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P} = \underline{\tau}_B^{ext}$
3. Work-Energy Principle: $T_1 + V_1^{ext} = T_2 + V_2^{ext}$ (All external forces are potential or do no work or no relative motion in line joining centers of particles)

Now for rigid bodies:

Linear Momentum Principle

$$\frac{d}{dt}\underline{P} = \underline{F}^{ext}$$
$$\underline{P} = M\underline{v}_c$$

M: Total Mass
$\underline{v}_c$: Velocity of center of mass

Angular Momentum Principle

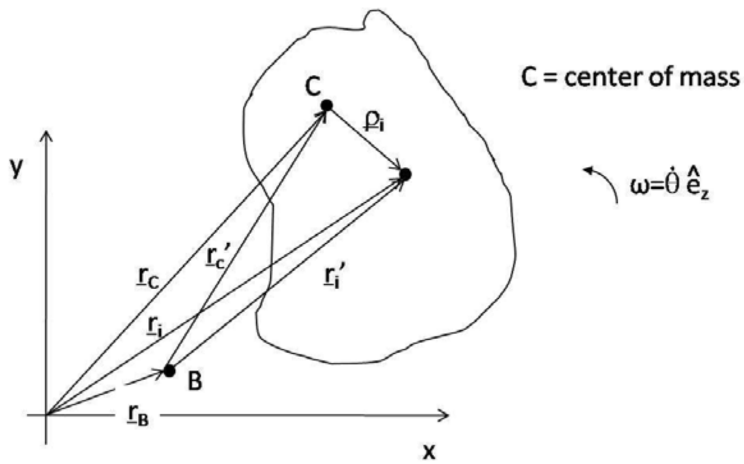

Figure 5: Rigid body that rotates with angular velocity ω. Figure by MIT OCW.

$$
\begin{aligned}
\underline{H}_B &= \sum_i (\underline{h}_B)_i = \sum_i \underline{r}_i^{'} \times \underline{p}_i \\
&= \sum_i \underline{r}_i^{'} \times m_i (v_c + \underline{\omega} \times \underline{\rho}_i) \\
&= \sum_i (\underline{r}_C^{'} + \underline{\rho}_i) \times (\underline{v}_c + \underline{\omega} \times \underline{\rho}_i)
\end{aligned} \tag{3}
$$

$$\underline{H}_B = \sum_i m_i (\underline{r}_C^{'} \times \underline{v}_C) + \sum_i (\underline{r}_C^{'} \times \underline{\omega} \times \underline{\rho}_i) + \sum_i m_i (\underline{\rho}_i \times \underline{v}_C) + \sum_i m_i (\underline{\rho}_i \times \underline{\omega} \times \underline{\rho}_i)$$

$\sum_i (\underline{r}_C^{'} \times \underline{\omega} \times \underline{\rho}_i) = 0$ because $\sum m_i \rho_i = 0$
$\sum_i m_i (\underline{\rho}_i \times \underline{v}_C) = 0$ because $\sum m_i \rho_i = 0$

$$\underline{H}_B = r_C^{'} \times \underline{P} + \sum_i m_i \underline{\rho}_i \times \underline{\omega} \times \underline{\rho_i}$$

$\sum_i m_i \underline{\rho}_i \times \underline{\omega} \times \underline{\rho_i}$: Contains *moment of inertia*.