2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 14

Lagrangian Dynamics: Virtual Work and Generalized Forces

Reading: Williams, Chapter 5

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

All q_i are scalars.
q_i: Generalized Coordinates
L: Lagrangian
Q_i: Generalized Forces

Admissible Variations/Virtual Displacements

Virtual Displacement:
Admissible variations: hypothetical (not real) small change from one geometrically admissible state to a nearby geometrically admissible state.

Bead on Wire

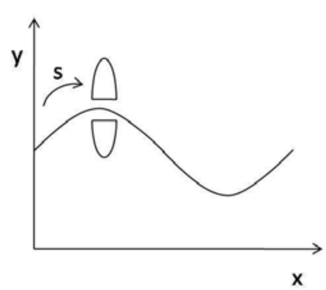

Figure 1: Bead on a wire. Figure by MIT OCW.

Both δ_x and δ_y are admissible variations. Hypothetical geometric configuration displacement.

$$\delta \neq d$$
$$\delta x \neq dx$$

dx implies t involved.

$$y = f(x)$$
$$dy = \frac{df}{dx} \cdot dx$$
$$\delta y = \frac{df(x)}{dx} \cdot \delta x$$

Generalized Coordinates

Minimal, complete, and independent set of coordinates

s is referred to as *complete*: capable of describing all geometric configurations at *all* times.
s is referred to as *independent*: If all but one coordinate is *fixed*, there is a continuous range of values that the free one can take. That corresponds to the admissible system configurations.

Example: 2-Dimensional Rod

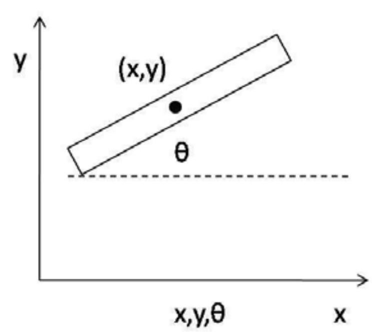

Figure 2: 2D rod with fixed translation in x and y but free to rotate about θ. Figure by MIT OCW.

If we fix x and y, we can still rotate in a range with θ.

degrees of freedom = # of generalized coordinates: True for 2.003J. True for Holonomic Systems.

Lagrange's equations work for Holonomic systems.

Virtual Work

$$W = \sum_i \underline{f}_i \cdot \underline{dr}_i \leftarrow \text{ Actual Work}$$

i = forces act at that location

$$\delta W = \sum_i \underline{f}_i \cdot \delta \underline{r}_i \leftarrow \text{ Virtual Work}$$

$$\underline{f}_i = \underline{f}_i^{\text{applied}} + \underline{f}_i^{\text{constrained}}$$

Constrained: Friction in roll. Constraint to move on surface. Normal forces. Tension, rigid body constraints.

$$\delta w = \sum_i \underline{f}_i^{\text{app}} \cdot \delta \underline{r}_i = 0 \text{ at equilibrium}$$

No work done because no motion in direction of force. No virtual work.

$$\sum_i \underline{f}_i = 0$$

Example: Hanging Rigid Bar

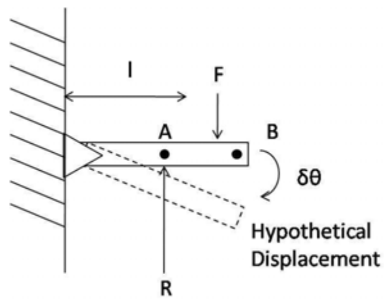

Figure 3: Hanging rigid bar. The bar is fixed translationally but is subject to a force, F. It therefore can displace itself rotationally about its pivot point. Figure by MIT OCW.

Displacement:

$$\delta \underline{y}_A = -a\delta\theta\hat{\jmath}$$
$$\delta \underline{y}_B = -l\delta\theta\hat{\jmath}$$

Forces:

$$\underline{F} = -F\hat{\jmath}$$
$$\underline{R} = R\hat{\jmath}$$

Two forces applied: $i = 2$

$$\delta w = Fl\delta\theta - Ra\delta\theta = 0$$
$$R = \frac{Fl}{a} \text{ at equilibrium}$$

Could also have taken moments about O.

Example: Tethered Cart

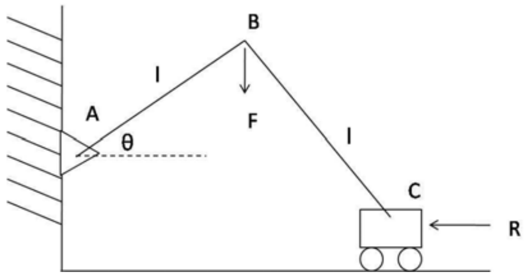

Figure 4: Tethered cart. The cart is attached to a tether that is attached to the wall. Figure by MIT OCW.

$$\delta w = F\delta y_B - R\delta x_c = 0$$

$$y_B = l\sin\theta$$

Using $\delta y = \frac{df(x)}{dx}\delta x_c$

$$\delta y_B = l\cos\theta\delta\theta$$

$$\delta x_c = -2l\sin\theta\delta\theta$$

$$(-Fl\cos\theta + 2R\sin\theta)\delta\theta = 0$$

$$-Fl\cos\theta + 2R\sin\theta = 0 \Rightarrow R = \frac{F}{2\tan\theta} \text{ at equilibrium}$$

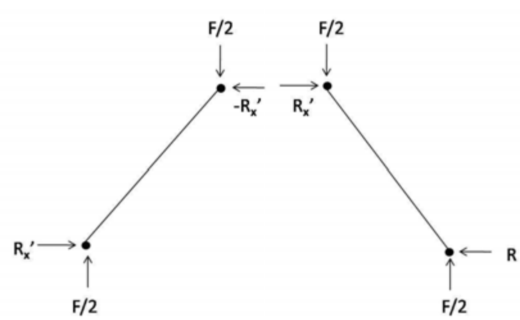

Figure 5: Application of Newton's method to solve problem. The indicated extra forces are needed to solve using Newton. Figure by MIT OCW.

Generalized forces for Holonomic Systems

In an *holonomic* system, the number of degrees of freedom equals the number of generalized coordinates.

$$\delta w = \sum_i \underline{f}_i \cdot \delta \underline{r}_i = \sum Q_i \delta q_j$$

i = number of applied forces: 1 to n
j = number of generalized coordinates

$$\underline{r}_i = r_i(q_1, q_2, \dots q_j)$$

r_i: Position of point where force is applied

$$\delta \underline{r}_i = \sum_j^m \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j$$

Substitute:

$$\sum_i^n \underline{f}_i \sum_j^m \frac{\partial \underline{r}_i}{\partial q_j} \cdot \delta q_j = \sum_j^m \left(\sum_i^n \underline{f}_i \frac{\partial \underline{r}_i}{\partial q_j} \right) \cdot \partial q_j$$

$$Q_j = \sum_i^n \underline{f}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} \text{ Generalized Forces}$$

$$\underline{f}_i = \underline{f}_i^{\text{NC}} + \underline{f}_i^{\text{CONS.}}$$

$f_i^{\text{CONS.}}$: Gravity, Spring, and Buoyancy are examples; Potential Function Exists.

$$\underline{f}^{\text{CONS.}} = -\frac{\partial V}{\partial \underline{r}}$$

Example:

$V_g = mgz$, $\underline{r} = z\hat{\jmath}$
$\underline{f}_g = -mg\frac{\partial z}{dz}\hat{\jmath} = -mg\hat{\jmath}$

$$f_i^{\text{cons.}} \cdot \frac{\partial \underline{r}_i}{\partial q} = -\frac{\partial V}{\partial \underline{r}} \frac{\partial \underline{r}}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

The conservative forces are already accounted for by the potential energy term in the Lagrangian for Lagrange's Equation

$$Q_j^{NC} = \sum_i^n \underline{f}_i^{NC} \cdot \frac{\partial \underline{r}_i}{\partial q_j}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{NC}}$$

Lagrange's Equation

Q_j^{NC} = nonconservative generalized forces

$\frac{\partial L}{\partial q_j}$ contains $\frac{\partial V}{\partial q_j}$.

Example: Cart with Pendulum, Springs, and Dashpots

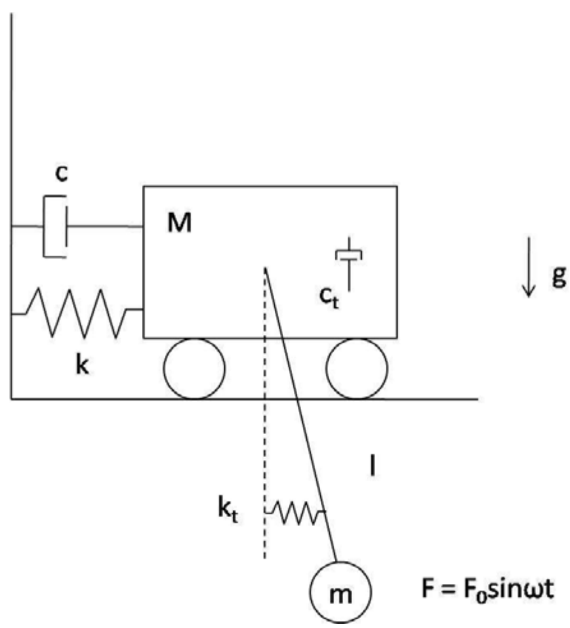

Figure 6: The system contains a cart that has a spring (k) and a dashpot (c) attached to it. On the cart is a pendulum that has a torsional spring (k_t) and a torsional dashpot (c_t). There is a force applied to m that is a function of time $F = F(t)$ We will model the system as 2 particles in 2 dimensions. Figure by MIT OCW.

4 degrees of freedom: 2 constraints. Cart moves in only 1 direction. Rod fixes distance of the 2 particles.

Thus, there are a net 2 degrees of freedom. For 2.003J, all systems are holonomic (the number of degrees of freedom equals the number of generalized coordinates).

$$q_1 = x$$
$$q_2 = \theta$$

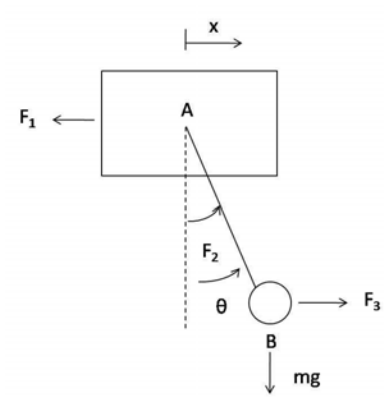

Figure 7: Forces felt by cart system. Figure by MIT OCW.

$\underline{F}_1$: Damper and Spring in $-x$ direction

$$-(kx + c\dot{x})\hat{\imath}$$

$\underline{F}_2$: Two torques:

$$\underline{\tau} = -(k_t\theta + c_t\dot{\theta}\hat{k}$$

$\underline{F}_3$:

$$\underline{F}_3 = F_0 \sin \omega t \hat{\imath}$$

$$\underline{r}_A = x\hat{\imath} = q_1\hat{\imath} \leftarrow \underline{r}_1$$

$$\underline{r}_B = \underline{r}_A + \underline{r}_{B/A} = (x + l\sin\theta)\hat{\imath} - l\cos\theta\hat{\jmath} \leftarrow \underline{r}_3$$

$$\underline{r}_2 = \theta\hat{k} \text{ (Torque creates angular displacement) } = q_2\hat{k}$$

$\underline{Q_1}$:

$\frac{\partial \underline{r}_1}{\partial q_1} = 1\hat{\imath}$, $\frac{\partial \underline{r}_2}{\partial q_1} = 0$, $\frac{\partial \underline{r}_3}{\partial q_1} = 1\hat{\imath}$

$$Q_1 = -c\dot{q}_1 + F_0 \sin \omega t$$

$\frac{\partial \underline{r}_1}{\partial q_2} = 0$, $\frac{\partial \underline{r}_2}{\partial q_2} = 1\hat{k}$, $\frac{\partial \underline{r}_3}{\partial q_2} = l \cos q_2 \hat{\imath} + l \sin q_2 \hat{\jmath}$

$$Q_2 = -c_t\dot{q}_2 + F_0 \sin \omega t \cdot l \cos q_2$$

With the generalized forces, we can write the equations of motion.

Kinematics

M:

$$\underline{r}_M = x\hat{\imath}$$
$$\dot{\underline{r}}_M = \dot{x}\hat{\imath}$$
$$\ddot{\underline{r}}_M = \ddot{x}\hat{\imath}$$

m:

$$\underline{r}_m = (x + l \sin \theta)\hat{\imath} - l \cos \theta \hat{\jmath}$$
$$\dot{\underline{r}}_m = (\dot{x} + l \cos \theta \cdot \dot{\theta})\hat{\imath} + l \sin \theta \dot{\theta} \hat{\jmath}$$
$$\ddot{\underline{r}}_m = (\ddot{x} + l(\cos \theta)\ddot{\theta} - l(\sin \theta)\dot{\theta}^2)\hat{\imath} + (l(\sin \theta)\ddot{\theta} + l(\cos \theta)\dot{\theta}^2)\hat{\jmath}$$

Generalized Coordinates: $q_1 = x$ and $q_2 = \theta$.

Lagrangian

$$L = T - V$$
$$T = T_M + T_m$$
$$T_M = \frac{1}{2}M(\dot{\underline{r}}_M \cdot \dot{\underline{r}}_M) = \frac{1}{2}M\dot{x}^2$$

$$T_m = \frac{1}{2}m(\dot{\underline{r}}_m \cdot \dot{\underline{r}}_m) \quad (1)$$
$$= \frac{1}{2}m(\dot{x}^2 + 2l\dot{x}\cos \theta \dot{\theta} + l^2\dot{\theta}^2) \quad (2)$$

$$\boxed{T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2l\dot{x}\cos\theta\dot{\theta} + l^2\dot{\theta}^2)}$$

$$V = V_{M,g} + M_{M,k} + V_{m,g} + V_{m,k_t} \tag{3}$$

$$= Mg(0) + \frac{1}{2}k(\underline{\dot{r}}_M \cdot \underline{\dot{r}}_M) + mg(-l\cos\theta) + \frac{1}{2}k_t\theta^2 \tag{4}$$

Symbol	Potential Energy
$V_{M,g}$	Gravity on M
$V_{M,k}$	Spring on M
$V_{m,g}$	Gravity on m
V_{m,k_t}	Torsional Spring on m

$$\boxed{V = \frac{1}{2}kx^2 + (-mgl\cos\theta) + \frac{1}{2}k_t\theta^2}$$

Substitute in $L = T - V$

$$\boxed{L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2) - \frac{1}{2}kx^2 + mgl\cos\theta - \frac{1}{2}k_t\theta^2}$$

Equations of Motion

Use $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = \Xi_i$ to derive the equations of motion. $\Xi_i = Q_i$.

From before, $\Xi_x = F_0 \sin\omega_0 t - c\dot{x}$ and $\Xi_\theta = F_0(\sin\omega t)l\cos\theta - c_t\dot{\theta}$.

For Generalized Coordinate x

$\delta x \neq 0$ and $\delta\theta = 0$. Units of Force.

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml(\cos\theta)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta + mL(-\sin\theta)\dot{\theta}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \boxed{(M+m)\ddot{x} + ml\ddot{\theta}(\cos\theta) + ml(-\sin\theta)\dot{\theta}^2 + kx = F_0\sin\omega t - c\dot{x}}$$

For Generalize Coordinate θ

$\delta x = 0$ and $\delta\theta \neq 0$. Units of Torque.

$$\frac{\partial L}{\partial \theta} = ml\dot{x}\dot{\theta}(-\sin\theta) - mgl\sin\theta - k_t\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{x}\cos\theta + ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\dot{x}(-\sin\theta)\dot{\theta} + ml\ddot{x}\cos\theta + ml^2\ddot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ml\dot{x}\dot{\theta}(-\sin\theta) + ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - ml\dot{x}\dot{\theta}(-\sin\theta) + mgl\sin\theta + k_t\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \boxed{ml\ddot{x}\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta + k_t\theta = F_0(\sin\omega t)l\cos\theta - c_t\theta}$$