2.004 MODELING DYNAMICS AND CONTROL II Spring 2003

Solutions for Problem Set 6

Problem 1. Rod sliding down wall. With the ends of the rod constrained to remain in contact with the wall and the floor, the rod is completely located by giving the angle θ between the rod and the vertical. Thus the system has one degree-of-freedom, and θ is a convenient generalized coordinate to describe the configuration of the system.

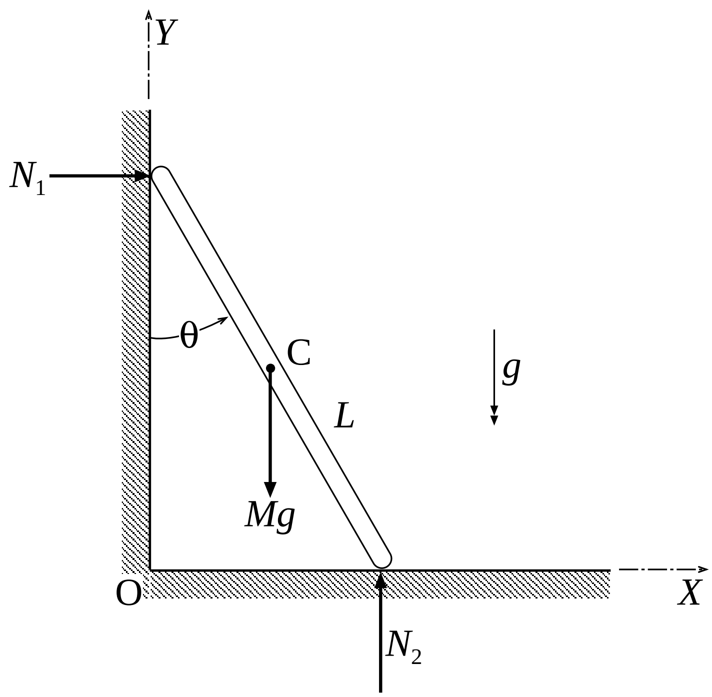

Figure 1: Free-body diagram with generalized coordinate θ.

The displacement components of the mass center C with respect to the reference frame XOY are

$$x = \frac{L}{2}\sin\theta \qquad \text{and} \qquad y = \frac{L}{2}\cos\theta$$

and the components of the linear velocity of the mass center are

$$\dot{x} = \frac{L}{2}\dot{\theta}\cos\theta \qquad \text{and} \qquad -\frac{L}{2}\dot{\theta}\sin\theta$$

The angular velocity of the rod is $\omega = \dot{\theta}$ in the counter-clockwise direction. The external forces acting on the rod are displayed in Fig.1: the resultant gravity force Mg, and the reactions from the wall and the floor. Since there is no friction, these reactions, N_1 and N_2, are *normal* to the surfaces. The equation of motion can be derived by applying the linear momentum principles in the X- and Y-directions and the angular momentum principle about the mass center C, and then eliminating the reaction forces N_1 and N_2 from these three equations by algebraic manipulation. This is the most straightforward procedure and will be described first. Afterwards, a second more sophisticated procedure, requiring only the single application of the angular momentum principle about a special moving point B, is described.

(i) Applying the linear momentum principle to the rod in Fig.1, one has

$$N_1 = \frac{d}{dt}(M\frac{L}{2}\dot{\theta}\cos\theta) = M\frac{L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \qquad (1)$$

in the X-direction, and

$$N_2 - Mg = \frac{d}{dt}(-M\frac{L}{2}\dot{\theta}\sin\theta) \qquad \text{or} \qquad N_2 = Mg - M\frac{L}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \qquad (2)$$

The moment of inertia of the rod about its center of mass C is $I_C = ML^2/12$. Applying the angular momentum principle about the mass center C yields

$$N_2\frac{L}{2}\sin\theta - N_1\frac{L}{2}\cos\theta = \frac{d}{dt}(M\frac{L^2}{12}\dot{\theta}) = M\frac{L^2}{12}\ddot{\theta} \qquad (3)$$

A single equation for θ is obtained by substituting N_1 from (1) and N_2 from (2) into (3) to get

$$\frac{L}{2}\sin\theta[Mg - M\frac{L}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)] - \frac{L}{2}\cos\theta[M\frac{L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)] = M\frac{L^2}{12}\ddot{\theta}$$

After cancelling the $\dot{\theta}^2$ terms and dividing through by ML^2 this reduces to

$$\left(\frac{1}{12} + \frac{1}{4}\sin^2\theta + \frac{1}{4}\cos^2\theta\right)\ddot{\theta} = \frac{g}{2L}\sin\theta \qquad \text{or} \qquad \ddot{\theta} - \frac{3}{2}\frac{g}{L}\sin\theta = 0 \qquad (4)$$

which is the desired equation of motion.

(ii) The second procedure requires a deeper study of the geometry and a familiarity with the generalized angular momentum equation

$$\sum \vec{\tau}_{ext} = \frac{d}{dt}(\vec{H}_B) + \vec{v}_B \times \vec{P} \qquad (5)$$

The advantage of the second procedure is that the desired equation of motion is obtained from a single application of (5).

As the rod AD of length L in Fig.3 slides down, it is always a diagonal of the rectangle ABDO. The nature of the motion is clarified when one notes that the other diagonal OB in the same rectangle also has the length L and rotates in the clockwise direction at the same rate $\omega = \dot{\theta}$ as the rod AD rotates in the counter-clockwise direction. The two diagonals have the same midpoint C. It thus becomes clear that the path of the rod's mass center C is a circular arc of radius $L/2$. The linear velocity of C is directed tengentially around this arc and has the magnitude $v_C = L\dot{\theta}/2$.

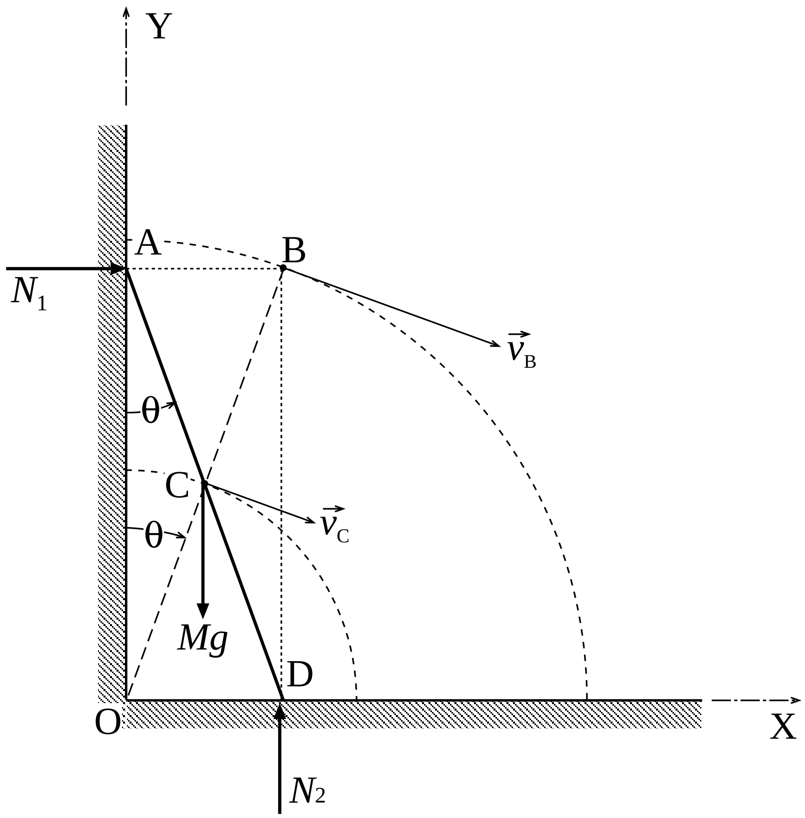

Figure 2: Study of the motions of points C and B.

Note also that the point B, which is at the intersection of the lines of action of the reaction forces N_1 and N_2, travels in a circle of radius L. The velocity of B is always parallel to the velocity of C and $\vec{v}_B = 2\vec{v}_C$.

The momentum requirements for this system are met, most economically, by applying the angular momentum principle about the moving point B. The reactions N_1 and N_2 have no moment about B. The angular momentum of the rod about B is

$$H_B = I_C\omega + \overrightarrow{BC} \times \vec{P} = M\frac{L^2}{12}\dot{\theta} + \frac{L}{2}\left(M\frac{L}{2}\dot{\theta}\right) = M\frac{L^2}{3}\dot{\theta}$$

The total external torque about B acting on the rod is due to gravity

$$\sum \tau_{ext} = Mg\frac{L}{2}\sin\theta$$

Then, since the velocity $\vec{v}_B$ of B is parallel to the linear momentum $\vec{P} = M\vec{v}_C$, the general angular momentum equation (5) reduces to

$$Mg\frac{L}{2}\sin\theta = \frac{d}{dt}\left(M\frac{L^2}{3}\dot{\theta}\right) \qquad \text{or} \qquad \ddot{\theta} - \frac{3}{2}\frac{g}{L}\sin\theta = 0$$

which is the desired equation of motion.

Problem 2. Unbalanced disk rolls down incline. The X- and Y-axes are set up with the X-axis along the inclined plane. The massless disk with the embedded particle m behaves like most rigid bodies: It has mass m, and a mass center C at the location of the particle, but because all the mass is concentrated at C, it has the special property that $I_C = 0$. The equation of motion will be derived by two different procedures. In the first method, two linear momentum equations and one angular momentum equation are used to eliminate the reaction forces N and T. In the second procedure the equation of motion is obtained directly from a single application of the angular momentum principle.

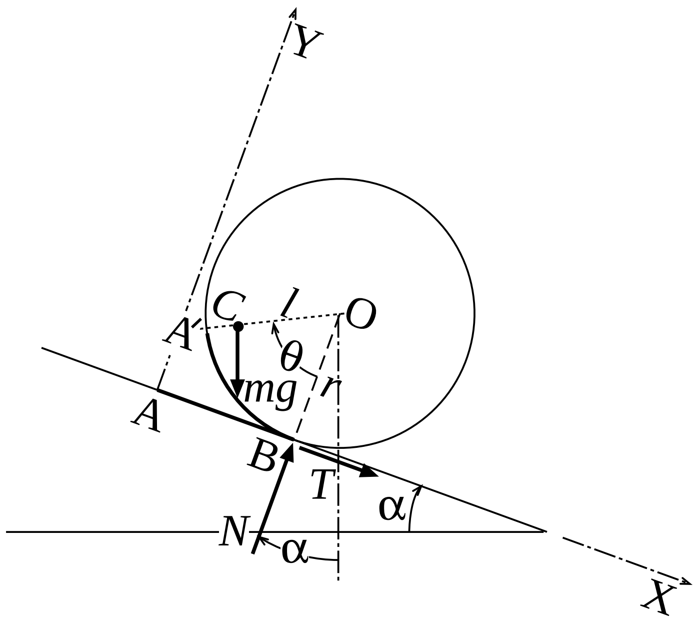

Figure 3: Free body diagram of disk rolling down incline.

(i) Studying the motion, we note that the distance AB rolled down the incline must be equal to the arc A′B $= r\theta$ if there is to be no slip. The x- and y-components of the displacement of the particle m, located at the point C, are

$$x = r\theta - l\sin\theta \qquad \text{and} \qquad y = r - l\cos\theta$$

and, by differentiation, the velocity components are

$$\dot{x} = r\dot{\theta} - l\dot{\theta}\cos\theta \qquad \text{and} \qquad \dot{y} = l\dot{\theta}\sin\theta$$

As a result the linear momentum components are

$$P_x = m(r\dot{\theta} - l\dot{\theta}\cos\theta) \qquad \text{and} \qquad P_y = ml\dot{\theta}\sin\theta$$

The forces acting on the disk are the gravity force mg and the reactions forces N and T displayed in Fig.3. The linear momentum principle requires

$$T + mg\sin\alpha = \frac{dP_x}{dt} = m(r\ddot{\theta} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta) \quad \text{or} \quad T = m(r\ddot{\theta} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta - g\sin\alpha) \tag{1}$$

in the x-direction and

$$N - mg\cos\theta = \frac{dP_y}{dt} = m(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta) \quad \text{or} \quad N = m(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta + g\cos\alpha) \tag{2}$$

in the y-direction. Because the moment of inertia about the point C vanishes, the angular momentum principle requires that the sum of the external torques about C must also vanish.

$$\sum \tau_C = Nl\sin\theta + T(r - l\cos\theta) = 0 \tag{3}$$

The three preceding equations are three equations for N, T, and θ. The equation of motion for θ is found by eliminating the reaction forces N and T. Inserting T from (1) and N from (2) into (3), one finds

$$m(l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta + g\cos\alpha)l\sin\theta + m(r\ddot{\theta} - l\ddot{\theta}\cos\theta + l\dot{\theta}^2\sin\theta - g\sin\alpha)(r - l\cos\theta) = 0$$

which simplifies to the desired equation of motion.

$$(r^2 + l^2 - 2rl\cos\theta)\ddot{\theta} + rl\dot{\theta}^2\sin\theta + g[l\sin(\alpha+\theta) - r\sin\alpha] = 0$$

(ii) In the second procedure, the reaction forces N and T do not enter at all because the angular momentum principle is applied about the contact point B. These forces are applied at B, so they have no torque about B. In order to apply the angular momentum principle about a moving point B, it is necessary to use the general vector statement of the angular momentum principle

$$\sum \vec{\tau}_B = \frac{d}{dt}(\vec{H}_B) + \vec{v}_B \times \vec{P} \tag{4}$$

(**Note on vector notation.** Vectors are commonly represented in documents by bold face **symbols**, by underlined symbols, or by arrows over symbols; *e.g.,* $\vec{f}$. For example, a vector velocity is represented by **v**, or $\underline{\mathrm{v}}$, or $\vec{v}$. In vector calculations it is convenient to use unit vectors aligned with a Cartesian coordinate system. These unit vectors are represented by the symbols **i**, **j**, and **k**, or by the symbols $\underline{\mathrm{e}}_x$, $\underline{\mathrm{e}}_y$, and $\underline{\mathrm{e}}_z$, or by the symbols $\vec{u}_x$, $\vec{u}_y$, and $\vec{u}_z$. The arrow-over notation is employed in these solutions.)

In the present problem, the only torque about B is due to the gravity force mg. From Fig.3 it is seen that the lever arm for the gravity force is the horizontal displacement between points C and B. This is the difference between the horizontal displacement of C from O $[l\sin(\alpha+\theta)]$ and the horizontal displacement of B from O $(r\sin\alpha)$. The counter-clockwise gravity torque is represented by the vector

$$\vec{\tau}_B = mg[l\sin(\alpha+\theta) - r\sin\alpha]\vec{u}_z \tag{5}$$

The angular momentum $\vec{H}_B$ of the particle m is the moment of the linear momentum $\vec{P}$ about B. Now

$$\vec{H}_B = \overrightarrow{\mathrm{BC}} \times \vec{P}$$

where

$$\overrightarrow{\mathrm{BC}} = -l\sin\theta\vec{u}_x + (r - l\cos\theta)\vec{u}_y \quad \text{and} \quad \vec{P} = P_x\vec{u}_x + P_y\vec{u}_y = m(r\dot{\theta} - l\dot{\theta}\cos\theta)\vec{u}_x + ml\dot{\theta}\sin\theta\vec{u}_y$$

so, (remember that the vector cross product of $\vec{A} = a_x\vec{u}_x + a_y\vec{u}_y$ and $\vec{B} = b_x\vec{u}_x + b_y\vec{u}_y$ is $\vec{A} \times \vec{B} = (a_x b_y - a_y b_x)\vec{u}_z$)

$$\vec{H}_B = -m[l^2 \sin^2\theta + (r - l\cos\theta)^2]\dot{\theta}\vec{u}_z = -m(r^2 + l^2 - 2rl\cos\theta)\dot{\theta}\vec{u}_z$$

and

$$\frac{d\vec{H}_B}{dt} = -m(r^2 + l^2 - 2rl\cos\theta)\ddot{\theta} + 2mrl\theta^2)\vec{u}_z \tag{6}$$

The last term in Eq.(4) is the cross product of the velocity $\vec{v}_B$ of the contact point B and the linear momentum $\vec{P}$ of the particle m embedded in the massless disk.

Since

$$\vec{v}_B = r\dot{\theta}\vec{u}_x \qquad \text{and} \qquad \vec{P} = m(r\dot{\theta} - l\dot{\theta}\cos\theta)\vec{u}_x + ml\dot{\theta}\sin\theta\vec{u}_y$$

the cross product is

$$\vec{v}_B \times \vec{P} = mrl\dot{\theta}^2 \sin\theta\vec{u}_z \tag{7}$$

Finally, inserting the terms (5), (6), and (7) into Eq.(4) the desired equation of motion is obtained.

$$(r^2 + l^2 - 2rl\cos\theta)\ddot{\theta} + rl\dot{\theta}^2\sin\theta + g[l\sin(\alpha + \theta) - r\sin\alpha] = 0$$

Problem 3. Mechanical model of a tight-rope walker.

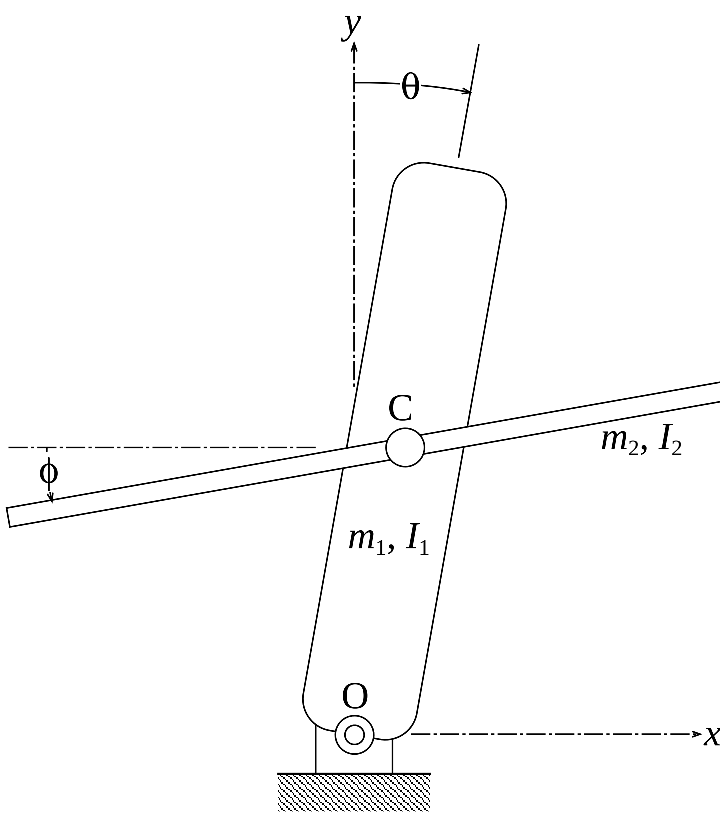

Figure 4: Mechanical model of tight-rope walker.

(a) The length OC is a. The coordinates of C in the frame Oxy are

$$x = a\sin\theta \qquad \text{and} \qquad y = a\cos\theta$$

Correct to first order in the small angle θ, the displacement and velocity components of C are

$$\begin{aligned} x &= a\theta \\ y &= a \end{aligned}$$

and

$$\begin{aligned} v_x &= \dot{x} = a\dot{\theta} \\ v_y &= \dot{y} = 0 \end{aligned}$$

(b) A free-body diagram of the rigid body representing the performer is shown in Fig.5.

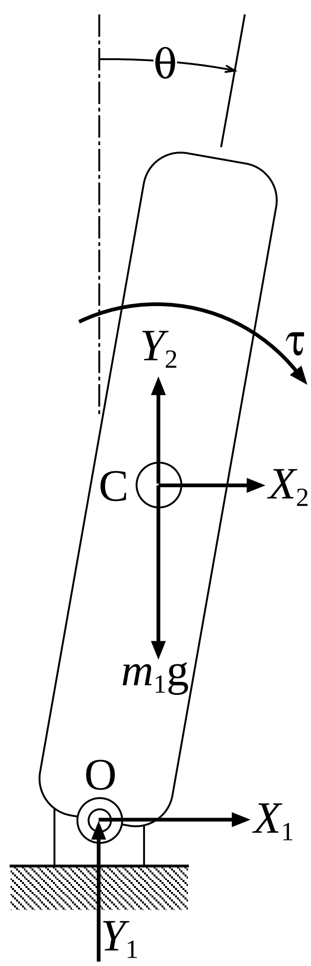

Figure 5: Free body diagram of performer model.

The reaction forces X_1 nd X_2 are from the pivot representing the tight rope. The reaction forces X_2 and Y_2 and the torque τ represent the actions of the balance pole acting on the performer. A free-body diagram of the balancing pole is shown in Fig.6.

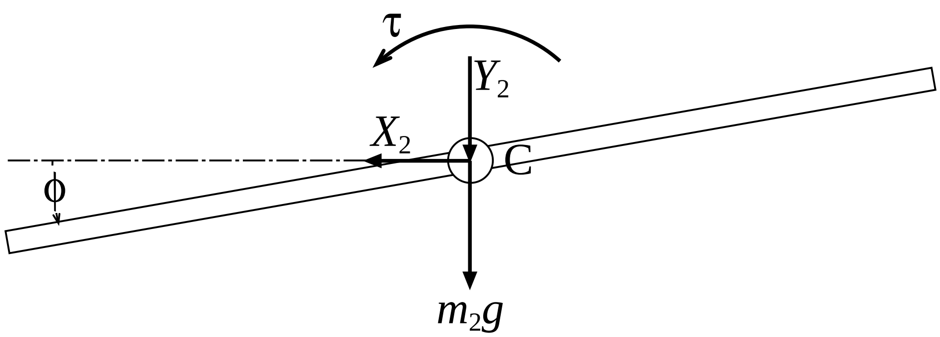

Figure 6: Free body diagram of balancing pole.

Note that the reaction forces and torque acting on the balancing pole from the performer are equal and opposite to the reaction forces and torque acting on the performer from the balance pole. This is required by Newton's third law of action and reaction.

(c) Applying the linear momentum principles to the rigid body representing the performer yields

$$\begin{aligned} X_1 + X_2 &= \frac{d}{dt}(m_1 a\dot{\theta}) \\ Y_1 + Y_2 - m_1 g &= 0 \end{aligned}$$

Applying the linear momentum principles to the balance pole yields

$$\begin{aligned} -X_2 &= \frac{d}{dt}(m_2 a\dot{\theta}) \\ -Y_2 - m_2 g &= 0 \end{aligned}$$

The reaction force components obtained from these four linear momentum equations are

$$\begin{aligned} Y_2 &= -m_2 g \qquad (1) \\ X_2 &= -m_2 a\ddot{\theta} \\ Y_1 &= (m_1 + m_2) g \\ X_1 &= (m_1 + m_2) a\ddot{\theta} \end{aligned}$$

(d) Applying the angular momentum principle about the mass center C of the balance pole yields

$$\tau = \frac{d}{dt}(I_2\dot{\phi}) = I_2\ddot{\phi} \qquad (2)$$

Applying the angular momentum principle about the mass center C of the rigid body representing the performer yields

$$\tau + Y_1 a\theta - X_1 a = \frac{d}{dt}(I_1\dot{\theta})$$

which, when the values of X_1 and X_2 given in (1) are inserted, reduces to

$$[I_1 + (m_1 + m_2)a^2]\ddot{\theta} - (m_1 + m_2)ga\theta = \tau \tag{3}$$

Equations (2) and (3) are independent equations of motion for $\phi(t)$ and $\theta(t)$.

An alternative procedure can be used to obtain equations of motion for θ and ϕ without introducing the reaction forces X_1, Y_1, X_2, and Y_2. The idea is to use angular momentum equations taken about points where these forces act. To eliminate the forces X_2 and Y_2, moments should be taken about the point C. A sub-system which doesn't introduce any other reaction forces is the balancing pole. Applying the angular momentum principle to the balancing pole about its mass center C produces a dynamic equation in which no reaction forces appear. This has, in fact, already been done in deriving Eq.(2). To eliminate the forces X_1 and Y_1, moments should be taken about the fixed point O. Here, in order to not reintroduce the internal reaction forces X_2 and Y_2, the angular momentum principle must be applied to the entire system of performer *and* balancing pole. The total clockwise torque about O acting on both performer and pole is

$$\tau_o = (m_1 + m_2)ga\theta$$

The total angular momentum about O in the clockwise direction is

$$H_o = (I_1 + m_1 a^2)\dot{\theta} - I_2\dot{\phi} + (m_2 a\dot{\theta})a$$

Since O is a fixed point the angular momentum equation has the form

$$\tau_o = \frac{dH_o}{dt} \qquad \text{or} \qquad (m_1 + m_2)ga\theta = [I_1 + (m_1 + m_2)a^2]\ddot{\theta} - I_2\ddot{\phi} \tag{4}$$

Equations (2) and (4) derived by this alternative procedure are a complete and independent set of equations of motion. Note that Eq.(4) is not identical to Eq.(3), but that the pair of (2) and (4) is equivalent to the pair of (2) and (3).