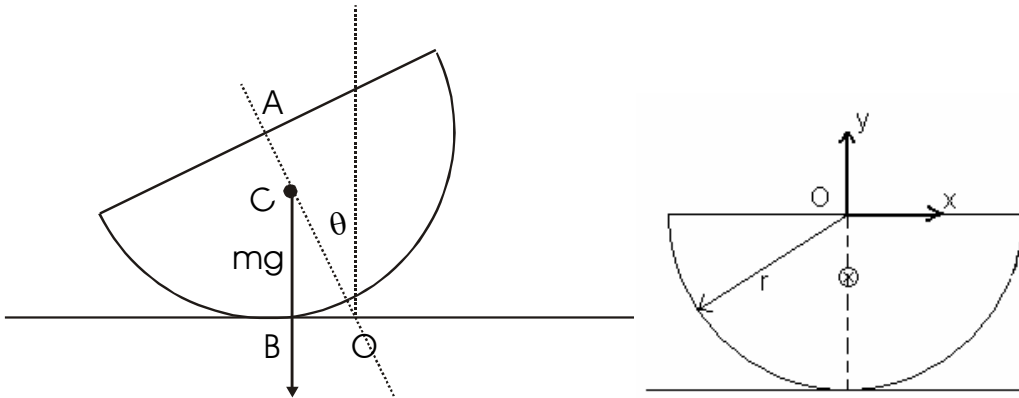


2.004: MODELING, DYNAMICS, & CONTROL II
Spring Term 2003

PLEASE ALSO NOTE THAT ALL PRELAB EXERCISE ARE DUE AT THE START (WITHIN 10 MINUTES) OF THE LAB SESSION, NO LATE WORK IS ACCEPTED.

Pre-Lab Exercise for Experiment 4

1. (a) Consider a “rocker”. The rocker consists of a half cylinder placed curve side down on a flat surface acted on by gravity. The geometric constraint is the “rolling without slipping” condition. Please calculate the center of mass of this system. Further calculate the moment of inertia of this rocker.



Location of ⊗:

$$\bar{x} = 0$$

by symmetry of resting figure on right.

$$\bar{y} = \frac{\int y^* dA}{A}$$

because of uniform mass properties

$$\bar{y} = \frac{\int_{-r}^0 y^* 2\sqrt{r^2 - y^2} dy}{\frac{\pi r^2}{2}}$$

the term inside the root is the x value at a given y, and 2 times that is the total width of the circle. the area is that of a semi-circle, rather than a full circle. and based on the origin defined in the figure on the right, we integrate from $y = -r$ to $y = 0$.

use u-substitution to solve this integral:

$$u = r^2 - y^2, \quad du = -2y dy, \quad \text{upper lim} = r^2, \quad \text{lower lim} = 0;$$

this gives us

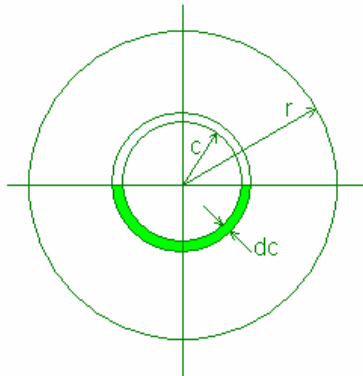
$$\bar{y} = \frac{\int_0^{r^2} \sqrt{u} du}{-\frac{\pi r^2}{2}}$$

$$\bar{y} = -\frac{2}{\pi r^2} * \frac{2}{3} * u^{\frac{3}{2}} \Big|_0^{r^2} = -\frac{4}{3\pi r^2} (r^3 - 0) = -\frac{4r}{3\pi} \quad \text{final answer}$$

moment of inertia of rocker:

first, find the moment of inertia about the origin, O, then use the parallel axis theorem to find it about the center of mass, \otimes .

$$I_{zzO} = \int_V \rho * c^2 dV$$



and dV is the volume of the lower half of the differential ring.

$dV = \frac{1}{2} (2\pi c dc dz)$ where dz is the differential width into the page.

the total width into the page will be denoted as l.

remember that mass $m = \rho * \text{volume} = \rho * \pi * r^2 * l / 2$

$$I_{zzO} = \int_0^l \int_0^r \rho * c^2 * \pi * c * dc dz$$

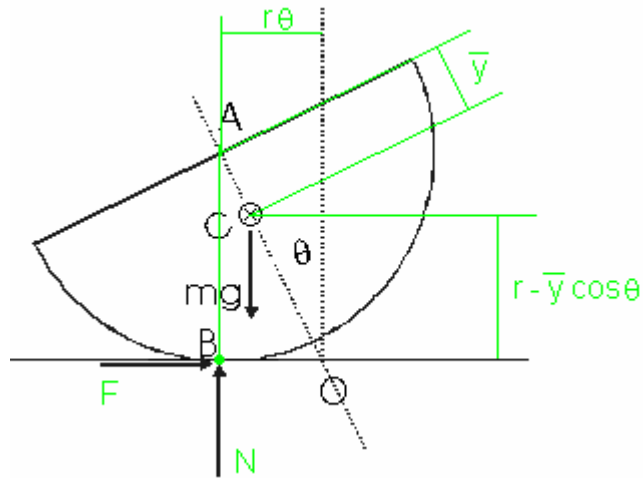
$$I_{zzO} = \rho \pi l \int_0^r c^3 dc = \rho \pi l \frac{r^4}{4} = \frac{m r^2}{2}$$

then use the parallel axis theorem to find $I_{zz\otimes}$

$$I_{zz\otimes} = I_{zzO} - m \bar{y}^2;$$

$$I_{zz\otimes} = 0.32 m r^2$$

(b) Write down the force and torque balance equations with reference to the center of mass.



measured from point O:

$$x = -r\theta + \bar{y} \sin\theta;$$

$$y = r - \bar{y} \cos\theta;$$

$$x' = -r\theta' + \bar{y} \cos\theta \theta';$$

$$y' = \bar{y} \sin\theta \theta';$$

$$x'' = -r\theta'' - \bar{y} \sin\theta \theta'^2 + \bar{y} \cos\theta \theta'';$$

$$y'' = \bar{y} \cos\theta \theta'^2 + \bar{y} \sin\theta \theta'';$$

$$\Sigma F_x: F = ma_x = m(-r\theta'' - \bar{y} \sin\theta \theta'^2 + \bar{y} \cos\theta \theta'');$$

$$\Sigma F_y: mg - N = ma_y = m(\bar{y} \cos\theta \theta'^2 + \bar{y} \sin\theta \theta'');$$

$$\Sigma \tau_O: F(r - \bar{y} \cos\theta) - N(\bar{y} \sin\theta) = I_{zzO} \theta'';$$

and torque about the contact point, B is

$$\Sigma \tau_B: -mg \bar{y} \sin\theta = I_{zzB} \theta_B'' \quad \text{where } I_{zzB} = -mr^2/2$$

(c) What is the natural frequency of the rocker for small angular oscillation around resting position?

using the torque equation and reshuffling terms:

$$\theta''(I_{zz\otimes} + mr^2 + m\bar{y}^2 - 2mr\bar{y}\cos\theta) + \theta'(mr\bar{y}\sin\theta) + mg\bar{y}\sin\theta = 0;$$

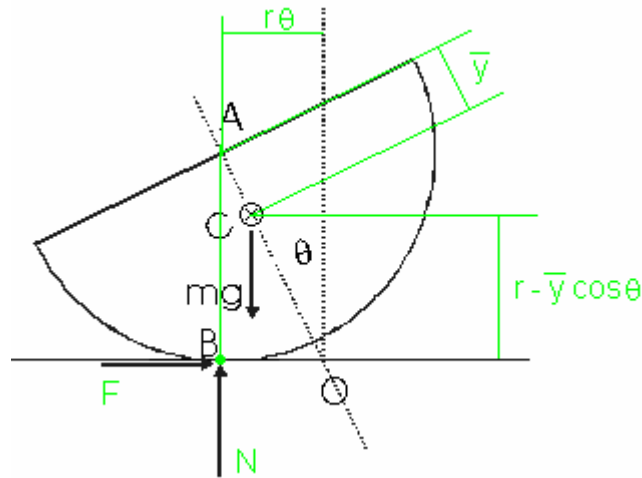
small angle approximation: $\cos\theta = 1$ and $\sin\theta = \theta$, θ'^2 very small.

$$\theta''(I_{zz\otimes} + mr^2 + m\bar{y}^2 - 2mr\bar{y}) + mg\bar{y}\theta = 0$$

then using similarity to second order equation $x'' + 2\xi\omega_n x' + \omega_n^2 = 0$;

$$\omega_n = \sqrt{\frac{g\bar{y}}{0.32r^2 + (r - \bar{y})^2}}$$

(d) Repeat previous calculation of the natural frequency by writing force and torque balance equations with reference to the moving contact point.



measured from point O:

$$\begin{aligned} x &= -r\theta; & y &= 0; \\ x' &= -r\theta'; & y' &= 0; \\ x'' &= -r\theta''; & y'' &= 0; \end{aligned}$$

$$\Sigma F_x: F = ma_x = -mr\theta'';$$

$$\Sigma F_y: mg - N = ma_y = 0;$$

$$\Sigma \tau_B: -mg \bar{y} \sin\theta = I_{zzB} \theta_B''$$

$$\text{where } I_{zzB} = I_{zzO} + m(\sqrt{(r - \bar{y} \cos \theta)^2 + (\bar{y} \sin \theta)^2})^2$$

$$I_{zzB} = 0.32mr^2 + m(r^2 - 2r\bar{y} \cos \theta + \bar{y}^2)$$

after small angle approximation ($\cos \theta = 1$), this becomes

$$I_{zzB} = 0.32mr^2 + m(r - \bar{y})^2$$

using the torque equation and reshuffling terms:

$$\theta''(0.32mr^2 + m(r - \bar{y})^2) + mg \bar{y} \sin\theta = 0;$$

small angle approximation: $\sin\theta = \theta$,

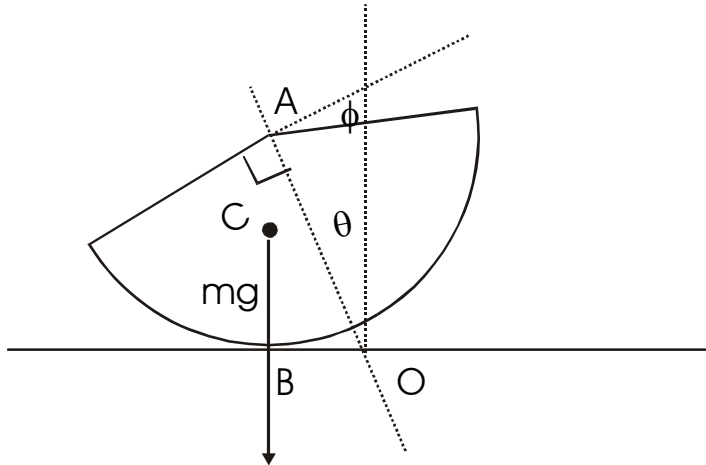
$$\theta''(0.32mr^2 + m(r - \bar{y})^2) + mg \bar{y} \theta = 0;$$

then using similarity to second order equation $x'' + 2\xi\omega_n x' + \omega_n^2 = 0$;

$$\omega_n = \sqrt{\frac{g\bar{y}}{0.32r^2 + (r - \bar{y})^2}}$$

Therefore, the natural frequency is same to the one obtained in part 1(c).

2. Consider an “asymmetric rocker.” Please keep the rolling without slipping boundary condition.



(a) Calculate the new center of mass. Further calculate the new resting angle of the rocker.

yeah, so this rocker isn't really asymmetric. it settles so that the resting angle is $\phi/2$. then the x-coord of the center of mass is 0 again, by symmetry.

Location of \otimes :

$\bar{x} = 0$ by symmetry of resting figure on right.

$\bar{y} = \frac{\int y * dA}{A}$ because of uniform mass properties

$$\bar{y} = \frac{\int_{-r}^{-r \sin(\phi/2)} y * 2\sqrt{r^2 - y^2} dy + \int_{-r \sin(\phi/2)}^0 y * 2 \cot(\phi/2) * y dy}{\frac{r^2(\pi - \phi)}{2}}$$

use u-substitution to solve the first integral, as before, and the second is simple. this gives:

$\bar{y} = -\frac{4r \cos(\phi/2)}{3(\pi - \phi)}$ final answer

moment of inertia of rocker:

first, find the moment of inertia about the origin, O, then use the parallel axis theorem to find it about the center of mass, \otimes .

$$I_{zz_o} = \int_V \rho * c^2 dV$$

dV is now $(2\pi - \pi - \phi)/2\pi * 2\pi c dc dz = (\pi - \phi)/2\pi * 2\pi c dc dz = (\pi - \phi)c dc dz$

remember that mass $m = \rho * \text{volume} = \rho * (\pi - \phi) * r^2 * l/2$

$$I_{zz_o} = \int_0^l \int_0^r \rho * c^2 * (\pi - \phi) * c * dc dz$$

$$I_{zz_o} = \rho(\pi - \phi)l \int_0^r c^3 dc = \rho(\pi - \phi)l \frac{r^4}{4} = \frac{mr^2}{2}$$

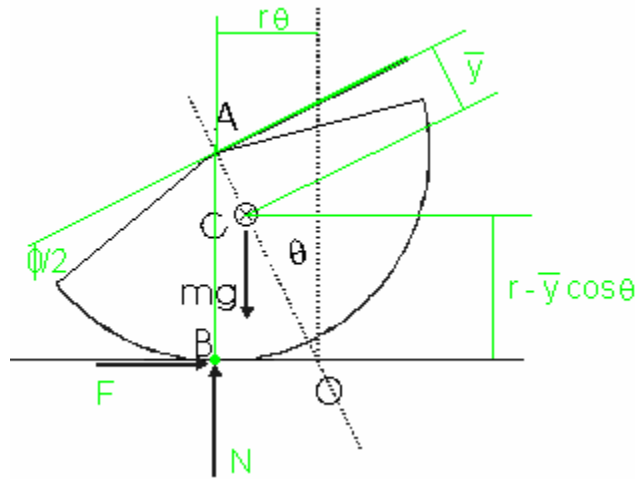
then use the parallel axis theorem to find $I_{zz\otimes}$

$$I_{zz\otimes} = I_{zz_o} - m \bar{y}^2;$$

$$I_{zz\otimes} = \left(0.5 - \frac{16r^2 \cos^2(\phi/2)}{9(\pi - \phi)^2} \right) mr^2$$

(b) Write down the force and torque balance equations at the center of mass.

The forces at and torque about the center of mass are the same as before, except that the value of m and \bar{y} in the force equations and I in the torque equation have changed.



$$\begin{aligned} x &= -r\theta + \bar{y} \sin\theta; & y &= r - \bar{y} \cos\theta; \\ x' &= -r\theta' + \bar{y} \cos\theta \theta'; & y' &= \bar{y} \sin\theta \theta'; \\ x'' &= -r\theta'' - \bar{y} \sin\theta \theta'^2 + \bar{y} \cos\theta \theta''; & y'' &= \bar{y} \cos\theta \theta'^2 + \bar{y} \sin\theta \theta''; \end{aligned}$$

$$\Sigma F_x: F = ma_x = m(-r\theta'' - \bar{y} \sin\theta \theta'^2 + \bar{y} \cos\theta \theta'');$$

$$\Sigma F_y: mg - N = ma_y = m(\bar{y} \cos\theta \theta'^2 + \bar{y} \sin\theta \theta'');$$

$$\Sigma \tau_{\odot}: F(r - \bar{y} \cos\theta) - N(\bar{y} \sin\theta) = I_{zz\odot} \theta'';$$

and torque about the contact point, B is

$$\Sigma \tau_B: -mg \bar{y} \sin\theta = I_{zzB} \theta_B'' \quad \text{where } I_{zzB} = -mr^2/2$$

(c) Calculate the natural frequency of the rocker for small angular oscillation around the resting angle.

ω_n is derived in the same way as in the semicircular case, but now I and \bar{y} have changed in magnitude.

$$\omega_n = \sqrt{\frac{g\bar{y}}{0.32r^2 + (r - \bar{y})^2}}$$

(d) Repeat previous calculation of the natural frequency by writing force and torque balance equations with reference to the moving contact point.

In the same way in part 1(d), natural frequency is same as the one obtained in part 2(c).