MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

**2.004 Dynamics and Control II**
Fall 2007

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Problem Set #8

**Solution** Posted: Problems 1–3: Friday, Nov. 9, '07

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In problems 1–3, you will explore the characteristics of the root locus. The root locus is the trajectory of the closed–loop pole as a gain $K$ increases. The closed–loop transfer function is $KG(s)/\left(1+KG(s)H(s)\right)$, and the closed–loop poles are the roots of $1+KG(s)H(s)=0$ (*i.e.*, the roots of the denominator of the closed–loop transfer function). In this problem set, we deal with unity–feedback only, which implies $H(s)=1$. Thus every pole on the root locus should satisfy $1+KG(s)=0$. Because $s$ is a complex number, $KG(s)=-1$ leads to two requirements:

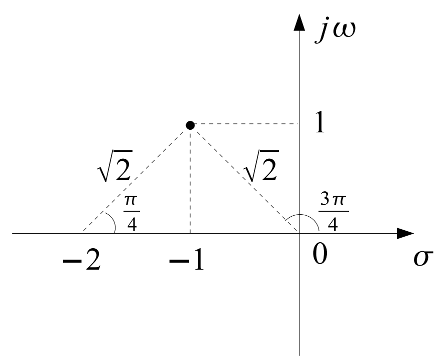
$$\begin{aligned} K &= 1/|G(s)|, \\ \angle KG(s) &= (2n+1)\times 180^\circ \quad \text{(n is an arbitrary integer).} \end{aligned}$$

You can use these relations either geometrically or algebraically.

**1.** For the complex number $s_1=-1+j$,

**a.** The phase of the complex number $(s_1+2)(s_1+0)$.

*Answer:*




From the above figure,

$$\begin{aligned} \angle(s_1+2)(s_1+0) &= \angle(s_1+2)+\angle(s_1+0) \\ &= \frac{\pi}{4}+\frac{3\pi}{4}=\pi. \end{aligned}$$

Algebraically,

$$(s_1+2)(s_1+0)=(-1+j+2)(-1+j)=(1+j)(-1+j)=-2,$$

and

$$\angle(-2) = \pi.$$

**b.** The value of the real number $K$ such that $K|s_1+2||s_1+0| = 1$.

*Answer:* From the above figure,

$$K \cdot \sqrt{2} \cdot \sqrt{2} = 1 \Rightarrow K = 1/2.$$

Algebraically,

$$|s_1+2| = |1+j| = \sqrt{1^2+1^2} = \sqrt{2}, \quad |s_1| = |-1+j| = \sqrt{1^2+1^2} = \sqrt{2}.$$

Hence,

$$K \cdot \sqrt{2} \cdot \sqrt{2} = 1 \Rightarrow K = 1/2.$$

**c.** Does $s_1$ belong to the root locus?

*Answer:* From the problem statement, the open–loop transfer function is given by

$$G(s) = \frac{1}{s(s+2)}.$$

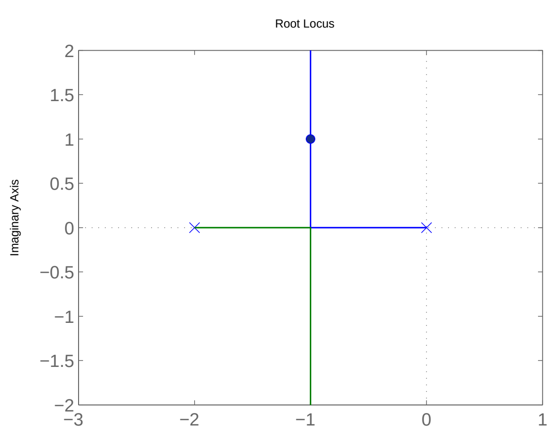
From result (a) we determine that $\angle G(s_1) = \pi$. Therefore, $s_1$ is on the root locus. To find the value of gain that would drive the closed–loop pole to location $s_1$ on the complex plane, we must satisfy

$$K\frac{1}{(s_1+0)\,(s_1+2)} = 1 \Rightarrow K\frac{1}{\sqrt{2}\sqrt{2}} = 1 \Rightarrow K = 2.$$

Note that result (b) is *not* directly applicable!

**d.** MATLAB result

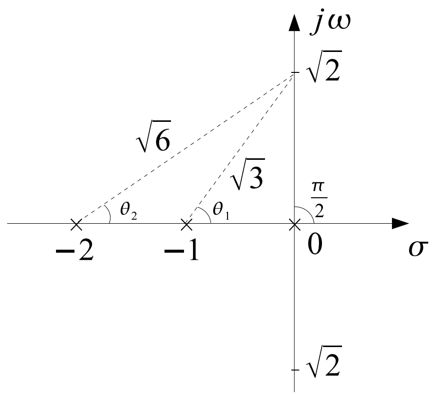
*Answer:*




**2.** The open loop transfer function $G(s) = 1/\left\{s(s+1)(s+2)\right\}$.

**a.** Show that $\pm j\sqrt{2}$ belongs to the root locus.

*Answer:*




For $s = \pm j\sqrt{2}$ to be on the root locus, it must satisfy $\angle \{s(s+1)(s+2)\} \Big|_{s=\pm j\sqrt{2}} = \pi$.

$$\angle \{s(s+1)(s+2)\} \Big|_{s=\pm j\sqrt{2}} =$$

$$\angle s\Big|_{s+\pm j\sqrt{2}} + \angle(s+1)\Big|_{s=\pm j\sqrt{2}} + \angle(s+2)\Big|_{s=\pm j\sqrt{2}} = \frac{\pi}{2} + \theta_1 + \theta_2 = \pi,$$

$$\Rightarrow \theta_1 + \theta_2 = \frac{\pi}{2},$$

where $\theta_1 = \tan^{-1}(\sqrt{2})$ and $\theta_2 = \tan^{-1}(\sqrt{2}/2)$. Hence, if $\theta_1 + \theta_2 = \pi/2$ so that $\cot(\theta_1 + \theta_2) = 0$, then $s = \pm j\sqrt{2}$ is on the root locus.

$$\cot(\theta_1 + \theta_2) = \frac{1}{\tan(\theta_1 + \theta_2)} = \frac{\cos(\theta_1 + \theta_2)}{\sin(\theta_1 + \theta_2)} = \frac{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}{\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2} = 0,$$

which requires that $\cos\theta_1 \cos\theta_2 = \sin\theta_1 \sin\theta_2$. From the geometric relation,

$$\cos\theta_1 \cos\theta_2 = \left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right) = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{6}}\right) = \sin\theta_1 \sin\theta_2,$$

which is true in this case. Therefore $s = \pm j\sqrt{2}$ belongs to the root locus.

(Or you can simply compute these angles with a calculator and verify that $\theta_1 + \theta_2 = \pi/2$.)

**b.** Compute the feedback gain $K$.

*Answer:* On the root locus, $K = 1/|G(s)|$. Hence,

$$K = \frac{1}{|G(s)|}\Big|_{s=\pm j\sqrt{2}} = \sqrt{2}\sqrt{3}\sqrt{6} = 6.$$

**c.** Verify algebraically.

*Answer:* If $s = \pm j\sqrt{2}$ belongs to the root locus, then it should satisfy $1 + KG(s) = 0$.

$$1 + K\frac{1}{s(s+1)(s+2)}\bigg|_{s=\pm j\sqrt{2}} = 1 + K\frac{1}{(\pm j\sqrt{2})(1 \pm j\sqrt{2})(2 \pm j\sqrt{2})} =$$

$$1 + K\frac{1}{\pm j\sqrt{2}(\pm j3\sqrt{2})} = 1 + K\frac{1}{-6} = 0.$$

Thus $K = 6$.

**d.** What will happen if $K$ exceeds the value that you computed in question (b)?

*Answer:* If $K > 6$, then the poles cross over to the right–hand half–plane. The system becomes unstable.

**e.** Sketch the root locus.

*Answer:*

- Since we have three poles $\Rightarrow$ the RL has 3 branches
- The RL has two real–axis segments: one between $s = 0$ and $s = -1$, the other one between $s = -2$ and negative infinity. You would expect to have a breakaway point between $s = 0$ and $s = -1$ since these are both real poles and a RL real–axis segment lies between them.
- The asymptotes: The system has three finite poles and no finite zero. Thus you would expect three zeros at infinity, which means that the RL must have three asymptotes.

$$\begin{aligned} \sigma_a &= \frac{-2-1}{3-0} = -1, \\ \theta_a &= \frac{(2m+1)\pi}{3-0} = \left\{\frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}\right\}. \end{aligned}$$

- Locating the break–in/away points:

$$K = -\sigma(\sigma+1)(\sigma+2),$$

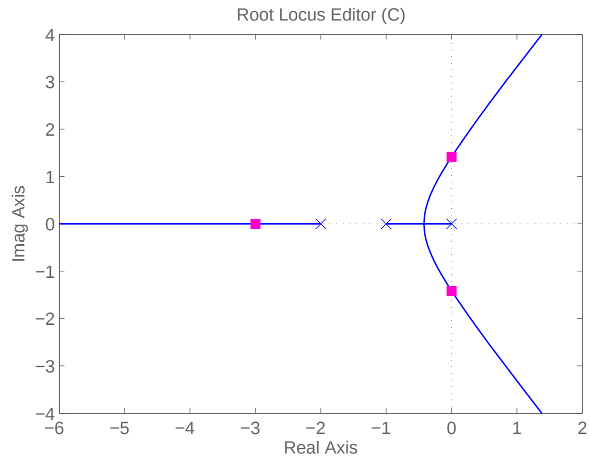
$$\frac{dK}{d\sigma} = -(\sigma+1)(\sigma+2) - \sigma(\sigma+2) - \sigma(\sigma+1) =$$
$$-\left[\sigma^2 + 3\sigma + 2 + \sigma^2 + 2\sigma + \sigma^2 + \sigma\right] = -\left(3\sigma^2 + 6\sigma + 2\right) = 0$$

$$\sigma = \frac{-3 \pm \sqrt{9 - 2\times 3}}{3} = \frac{-3 \pm \sqrt{3}}{3}.$$

Since there is no real–axis segment between $s = -2$ and $s = -1$, $s = \frac{-3-\sqrt{3}}{3}$ is not a break–in/away point. $s = \frac{-3+\sqrt{3}}{3}$ is break–away point because it lies between two poles ($s = 0$ and $s = -1$). We had expected a break–away point somewhere in this segment (see bullet #2 above.)

- *check the exact plot by* MATLAB *in (f).*

**f.** MATLABroot locus.




**3.** The open loop transfer function $G(s) = 1/\{(s+1)(s+2)\}$.

**a.** Sketch the root locus

*Answer:*

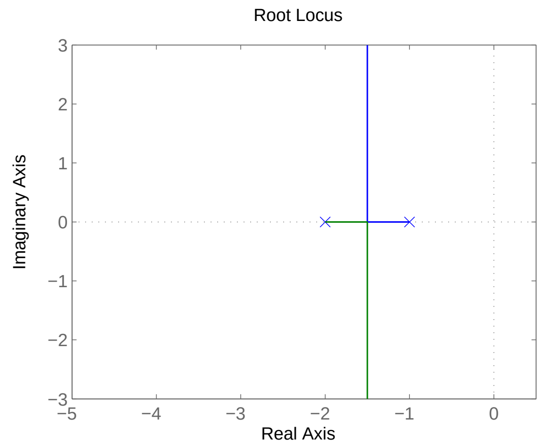
- 2 poles $\Rightarrow$ 2 branches
- One real–axis segment exists in between $s = -1$ and $s = -2$. We expect a break–away point between these two poles
- Asymptotes: The system has two finite poles and no finite zero. Thus you would expect two zeros at infinity, which means the RL has two asymptotes.

$$\sigma_a = \frac{-1-2}{2-0} = -\frac{3}{2},$$
$$\theta_a = \frac{(2m+1)\pi}{2-0} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}.$$

- Break–in/away points

$$K = -(\sigma+1)(\sigma+2) = 0,$$
$$\frac{dK}{d\sigma} = -(\sigma+2) - (\sigma+1) = -2\sigma - 3 = 0,$$
$$\Rightarrow \sigma = -\frac{3}{2}.$$

- The breakaway point and asymptotes' real–axis intercept $\sigma_a$ coincide with each other. Therefore, the RL lies exactly on top of the asymptote.




**b.** The closed–loop poles that yield 16.3% OS.

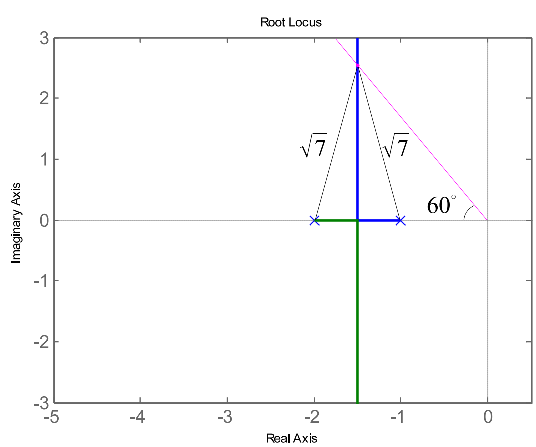
*Answer:*

The damping ratio $\zeta$ that yields 16.3% OS is computed by

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} = 0.5.$$

$$\cos\theta = 0.5 \Rightarrow \theta = 60^\circ.$$

You draw the line whose angle is $60^\circ$ to the negative real–axis; the intersection between this line and the root locus is the closed–loop pole that yields 16.3% OS. From the root locus, the real part of the pole is $-2$. Hence, geometrically $p_0 = -3/2 + j3/2 \times \tan(60^\circ) = -3/2 + j3\sqrt{3}/2$. Since the system is second order, this answer is exact.



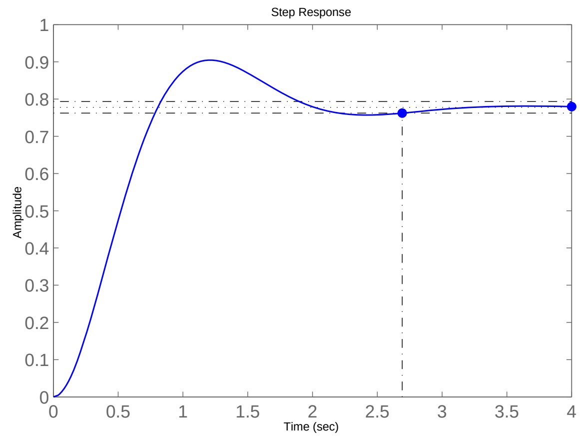

**c.** The settling time.

*Answer:* From the previous result in (b), the absolute value of the real–part of the pole is $\sigma_d = \zeta\omega_n = 3/2$ and the imaginary part is $\omega_d = \omega_n\sqrt{1-\zeta^2} =$

$3\sqrt{3}/2$. Since the settling time $T_s \approx 4/(\zeta\omega_n)$, you find

$$T_s \approx \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d} = \frac{4}{3/2} = \frac{8}{3}.$$

Note that the computed settling time agrees with the MATLABresult.




**d.** Using geometrical arguments and calculations, compute the value of the gain $K$.

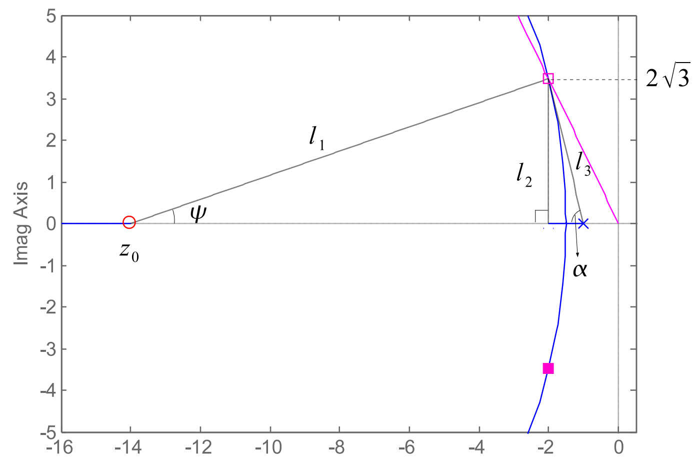
*Answer:* From the figure drawn in question (c), $|p_0+2| = \sqrt{7}$ and $|p_0+1| = \sqrt{7}$. Thus we find

$$K = |p_0+2||p_0+1| = \sqrt{7} \times \sqrt{7} = 7.$$

**e.** Using a PD controller, achieve the settling time to 75% of the value obtained in question (c) while maintaining the same overshoot.

*Answer:* The new settling time is $2 (= 8/3 \times 75\%)$ (s). Thus the absolute value of the real part $\sigma'_d$ of the new pole is 2. To maintain the same %OS, the pole should have the same damping ratio, which means the angle $\theta$ remains the same. So the imaginary part $\omega'_d$ of the pole should be $2 \times \tan(60°) = 2\sqrt{3}$.

The new pole is at $s_1 = -2 + j2\sqrt{3}$. To achieve a RL that overlaps the $s_1$ location, we cascade to the open–loop TF a zero at $s = z_0$ (cascading a zero to the open–loop TF constitutes the PD controller.) Now we must determine the location $z_0$ of the zero.




Denoting $\angle(s_1 + z_0) = \psi$, $|s_1 + z_0| = l_1$, $\angle(s_1 + 2) = \pi/2$, $|s_1 + 2| = l_2$, $\angle(s_1 + 1) = 180 - \alpha$, and $|s_1 + 1| = l_3$, we want to make $s_1$ belong to the root locus. First we apply $\angle KG(s) = 180°$.

$$90° + (180° - \alpha) - \psi = 180° \quad \Rightarrow \psi = 90° - \alpha.$$

From the geometry,

$$\tan\alpha = \frac{2\sqrt{3}}{1} = 2\sqrt{3}.$$

Thus $\alpha = 73.8979°$. Substituting it in $\psi = 90° - \alpha$, we find

$$\psi = 90° - 73.8979° = 16.1021°.$$

From the geometry,

$$\tan\psi = \frac{2\sqrt{3}}{z_0 - 2} \Rightarrow z_0 = 2 + \frac{2\sqrt{3}}{\tan\psi} = 2 + \frac{2\sqrt{3}}{0.2887} = 14.$$

Thus, the requisite PD compensator is $(s + 14)$. To find the gain $K$,

$$\frac{Kl_1}{l_2 l_3} = 1 \Rightarrow K = \frac{l_2 l_3}{l_1} = \frac{2\sqrt{3}\sqrt{1 + (2\sqrt{3})^2}}{\sqrt{12^2 + (2\sqrt{3})^2}} = \frac{2\sqrt{3}\sqrt{13}}{\sqrt{156}} = 1.$$

Note that MATLAB's `sisotool` uses different notation for the compensator, you have to re–calculate a gain $K_z$ to use MATLAB's `sisotool` as follows:

$$K(s + z_0) = K_z(\frac{s}{z_0} + 1) \Rightarrow K_z = 14.$$

**f.** Sketch the root locus of the PD-compensated system.

*Answer:*

- 2 poles $\Rightarrow$ 2 branches
- Two real–axis segments: one in between $s = -1$ and $s = -2$, and the other in between $s = -14$ and negative infinity. We expect a break–away point between the two poles at $-1$, $-2$.
- Asymptotes: The system has two finite poles and one zero. You would expect one zero at infinity, which means that the RL has one asymptote.

$$\theta_a = \frac{(2m+1)\pi}{2-1} = \{\pi\}\,.$$

  So the asymptote is the part of the real axis towards $-\infty$.
- The break–in/away points

$$K = -\frac{(\sigma+1)(\sigma+2)}{\sigma+14},$$

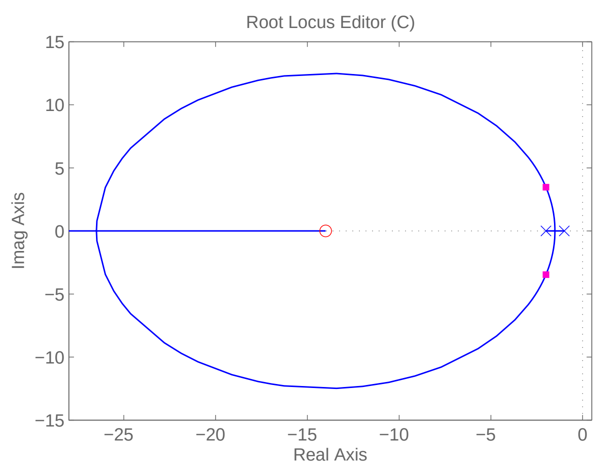
$$\frac{dK}{d\sigma} = -\frac{(2\sigma+3)(\sigma+14)-(\sigma^2+3\sigma+2)}{(\sigma+14)^2} =$$
$$\frac{(2\sigma^2+31\sigma+42)-(\sigma^2+3\sigma+2)}{(\sigma+14)^2} = \frac{\sigma^2-28\sigma+40}{(\sigma+14)^2} = 0,$$

$$\sigma^2 - 28\sigma + 40 = 0$$
$$\sigma = 14 \pm \sqrt{14^2-40} = 14 \pm \sqrt{156} = \{26.49,\ 1.51\}$$

The first solution is a break–in point (because it lies on the real axis segment of the RL between the zero at $-14$ and the zero at infinity) and the second solution is a break–away point (because it lies on the real axis segment of the RL between the poles at $-1$, $-2$.)

**g.** Verify numerically using MATLAB.

*Answer:*



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**Solution** Posted: Problems 4–5: Wednesday, Nov. 14, '07

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**4.** Matrices algebra.

*Answer:*

- $s\mathbf{I} - \mathbf{A}$

$$s\mathbf{I} - \mathbf{A} = s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} s-3 & -1 \\ 1 & s-3 \end{pmatrix}$$

- $\mathbf{AB}$

$$\mathbf{AB} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 7 & 16 \end{pmatrix}$$

Note that

$$\mathbf{BA} = \begin{pmatrix} 7 & -1 \\ 4 & 18 \end{pmatrix} \neq \mathbf{AB}.$$

Matrices do not commute.

- $\mathbf{B}^{-1}\mathbf{A}$

$$\mathbf{B}^{-1} = \frac{1}{2\cdot 5 - 3\cdot(-1)}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix},$$

$$\mathbf{B}^{-1}\mathbf{A} = \frac{1}{13}\begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 14 & 8 \\ -11 & 3 \end{pmatrix}.$$

- $\mathbf{Bp}$

$$\mathbf{Bp} = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

- $\mathbf{Aq}$

$$\mathbf{Aq} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} e^{-3t}\cos t \\ e^{-3t}\sin t \end{pmatrix} = \begin{pmatrix} 3e^{-3t}\cos t + e^{-3t}\sin t \\ -e^{-3t}\cos t + 3e^{-3t}\sin t \end{pmatrix}.$$

**5.** The *compensated* 2.004 Tower system.

**a)** Forces acting on the tower.

*Answer:*

- Inertia force: $-m_1\ddot{x}_1(t)$
- Spring force: $-k_1x_1(t) + k_2(x_2(t) - x_1(t))$
- Damping force: $-b_1\dot{x}_1(t) + b_2(\dot{x}_2(t) - \dot{x}_1(t))$
- Wind force: $w(t)$
- Actuator force: $-a(t)$

Applying force balance, we obtain an equation of motion for the tower.

$$m_1\ddot{x}_1(t) + (b_1 + b_2)\dot{x}_1(t) - b_2\dot{x}_2(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = w(t) - a(t).$$

**b)** Forces acting on the slider.

*Answer:*

- Inertia force: $-m_2\ddot{x}_2(t)$
- Spring force: $-k_2(x_2(t) - x_1(t))$
- Damping force: $-b_2(\dot{x}_2(t) - \dot{x}_1(t))$
- Actuation force: $a(t)$

Applying force balance, we obtain an equation of motion for the slider.

$$m_2\ddot{x}_2(t) + b_2(\dot{x}_2(t) - \dot{x}_1(t)) + k_2(x_2(t) - x_1(t)) = a(t).$$

**c)** The equations of motion in terms of the state variables.

*Answer:* Setting four state variables $\{x_1, v_1, x_2, v_2\}$ and omitting time dependency $(t)$ for simplicity, we can re–write the equations of motion as follows:

$$\begin{aligned} m_1\dot{v}_1 + (k_1 + k_2)x_1 + (b_1 + b_2)v_1 - k_2x_2 - b_2v_2 &= -a + w, \\ m_2\dot{v}_2 - k_2x_1 - b_2v_1 + k_2x_2 + b_2v_2 &= a, \end{aligned}$$

Substituting $\{x_1, v_1, x_2, v_2\}$ to $\{q_1, q_1, q_2, q_2\}$, we obtain the equations of motion for the state variables as follows:

$$\begin{aligned} m_1\dot{q}_2 + (k_1 + k_2)q_1 + (b_1 + b_2)q_2 - k_2q_3 - b_2q_4 &= w - a, \\ m_2\dot{q}_4 - k_2q_1 - b_2q_2 + k_2q_3 + b_2q_4 &= a. \end{aligned}$$

**d)** Solve the equations of motion for $\dot{q}_2$ and $\dot{q}_4$.

*Answer:*

$$\begin{aligned} \dot{q}_2 &= -\frac{(k_1 + k_2)}{m_1}q_1 - \frac{(b_1 + b_2)}{m_1}q_2 + \frac{k_2}{m_1}q_3 + \frac{b_2}{m_1}q_4 - \frac{1}{m_1}a + \frac{1}{m_1}w, \\ \dot{q}_4 &= \frac{k_2}{m_2}q_1 + \frac{b_2}{m_2}q_2 - \frac{k_2}{m_2}q_3 - \frac{b_2}{m_2}q_4 + \frac{1}{m_2}a \end{aligned}$$

**e)** State–space representation

*Answer:*

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2)/m_1 & -(b_1+b_2)/m_1 & k_2/m_1 & b_2/m_1 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & b_2/m_2 & -k_2/m_2 & -b_2/m_2 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ -1/m_1 \\ 0 \\ 1/m_2 \end{pmatrix},$$

and

$$\mathbf{G} = \begin{pmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{pmatrix}.$$