Session 19

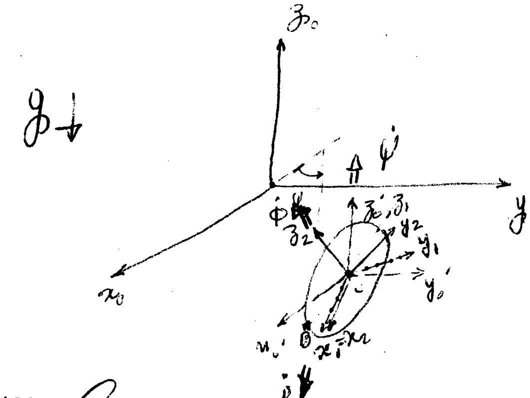
Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) - \frac{\partial L}{\partial q_j} = Q_j + \sum_{i=1}^{m} \lambda_i a_{ij} \qquad (1)$$

$i = 1, \dots, m$
$j = 1, \dots, n$

$$\sum_{j=1}^{n} a_{ij} \dot q_j + b_i = 0$$

Example: Use of Lagrangian multipliers for nonholonomic systems
rolling penny:
Thin disk rolling without slip on a horizontal plane



Initial choice of Coordinates
$(\underbrace{x, y, z}_{\text{position of } c},\ \underbrace{\psi, \nu, \phi}_{\text{Euler angles of "3-1" type}})$

(*) Constraints: $\underline{V}_B = \underline{0} \Rightarrow$ 3 Constraints $\Rightarrow$ # DOF $= 6-3=3$

$\underline{v}_B = \underline{v}_C + \underline{\omega} \times \underline{r}_{CB}$; $\underline{V}_C = \dot x\, \underline{i} + \dot y\, \underline{j} + \dot z\, \underline{K}$, (i, j, k are unit Vectors in x_0', y_0', z_0' frame)

$$\omega = \dot\psi + \dot\nu + \dot\phi$$
$$= \dot\nu\, \hat i_2 + \dot\psi \sin\nu\, \hat j_2 + (\dot\psi \cos\nu + \dot\phi)\hat k_2$$

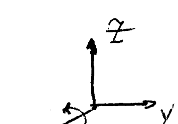
Here (i_2, j_2, k_2) are unit Vectors in the (x_2, y_2, z_2) frame)

Also $\underline{r}_{CB} = -R\, j_2$

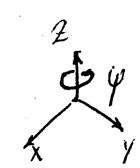
$$\underline{\omega} \times \underline{r}_{CB} = R(\dot\psi \cos\nu + \dot\phi)\, i_2 - R\dot\nu\, k_2$$

NOTE: $\underline{i}_2 = \underbrace{\underline{\underline{R}}_3\, \underline{\underline{R}}_1\, \underline{\underline{R}}_3^T}_{\text{Representation of "1" rotation in the } [i, j, k] \text{ frame}} \underline{\underline{R}}_3\, \underline{i} = \underline{\underline{R}}_3\, \underline{\underline{R}}_1\, \underline{i}$

where $\underline{\underline{R}}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\nu & -\sin\nu \\ 0 & \sin\nu & \cos\nu \end{pmatrix}$



$\underline{\underline{R}}_3 = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$



$$i_2 = \begin{pmatrix} \cos\psi \\ \sin\psi \\ 0 \end{pmatrix}; \quad K_2 = \begin{pmatrix} \sin\psi \sin\nu \\ -\cos\psi \sin\nu \\ \cos\nu \end{pmatrix} \quad (\underline{K}_2 = \underline{\underline{R}}_3 \underline{\underline{R}}_1 \underline{k})$$

(*) gives $i=1$ (a) $\dot{x} + \dot{\psi} R \cos\psi \cos\nu + \dot{\phi} R \cos\psi - \dot{\nu} R \sin\psi \sin\nu = 0$

$i=2$ (b) $\dot{y} + \dot{\psi} R \sin\psi \cos\nu + \dot{\phi} R \sin\psi - \dot{\nu} R \cos\psi \sin\nu = 0$ } non halonomic

(c) $\dot{z} - \dot{\nu} R \cos\nu = 0 \longrightarrow \boxed{z - R\sin\nu = 0}$ Halonomic

halonomic Constraint makes it possible to pass to the generalized Coordinates

$$(\underset{q_1}{x}, \underset{q_2}{y}, \underset{q_3}{\psi}, \underset{q_4}{\nu}, \underset{q_5}{\phi}) \quad (5)$$

Since the rolling Constraint is ideal eq (1) applies

$L = T - V$

$T = \frac{1}{2} m |V_c|^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{I}}_c \underline{\omega}$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} [I_{x2} \dot{\nu}^2 + I_{y2} \dot{\psi}^2 \sin^2\nu + I_{z2} (\dot{\psi}\cos\nu + \dot{\phi})^2]$

USE HOLONOMIC CONSTRAINT

$I_{x2} = I_{y2} = \frac{1}{4} m R^2 \quad I_{z2} = \frac{1}{2} m R^2$

$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{\nu}^2 R^2 \cos^2\nu) + \frac{1}{8} m R^2 [\dot{\nu}^2 + \dot{\psi}^2 \sin^2\nu + 2(\dot{\psi}\cos\nu + \dot{\phi})^2]$

$V = mgR \sin\nu$

$\Rightarrow L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{\nu}^2 R^2 \cos^2\nu) + \frac{1}{8} m R^2 [\dot{\nu}^2 + \dot{\psi}^2 \sin^2\nu + 2(\dot{\psi}\cos\nu + \dot{\phi})^2] - mgR\sin\nu = 0$

- $Q_j = 0. \quad j = 1, \dots, 5$

(No non-potential active forces)

Identify "a_{ij}": $\quad i = 1, \dots, 2 \quad j = 1, \dots, 5$

$a_{11} = 1 \quad a_{12} = 0 \quad a_{13} = R\cos\psi\cos\nu \quad a_{14} = -R\sin\psi\sin\nu \quad a_{15} = R\cos\psi$

$a_{21} = 0 \quad a_{22} = 1 \quad a_{23} = R\sin\psi\cos\nu$

$a_{24} = R\cos\psi\sin\nu \quad a_{25} = R\sin\psi$

Eq. of motion

(1) $m\ddot{x} = \lambda_1$

(2) $m\ddot{y} = \lambda_2$

(3) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = R\cos\nu(\lambda_1 \cos\psi - \lambda_2 \sin\psi)$

$\frac{\partial L}{\partial \varphi} = 0$: no dependence on φ, but since the problem is non-holonomic it doesn't help us — cyclic coordinate

(4) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\nu}}\right) - \frac{\partial L}{\partial \nu} = R\sin\nu(-\lambda_1 \sin\psi + \lambda_2 \cos\psi)$

(5) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) - \frac{\partial L}{\partial \phi} = R(\lambda_1 \cos\psi + \lambda_2 \sin\psi)$

+ Constraints (a) & (b)

In solving eqs, use (1) and (2) to eliminate λ_1 & λ_2 from the remaining equations

$\Rightarrow$ 5 ODE for five coordinates