Eq. of Small Oscillation

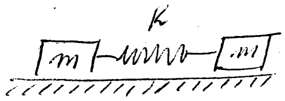
$$\underline{M}\,\ddot{\underline{x}} + \underline{K}\,\underline{x} = 0 \qquad \tilde{x}(t) = \underline{a}\,e^{\lambda t}$$

$$\underline{x} = q - q_0 \qquad\qquad = \underline{a}\,e^{\pm i\omega t}$$

K is positive def.

$$x(t) = \sum_j c_j^{\pm}\,\underline{a}_j\,e^{\pm i\omega t}$$

Example 2



Guess mode shapes $\quad a_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \omega_1^2 = \frac{2K}{m}$

$\underline{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \omega_2^2 = 0 \quad$ rigid body mode

For $\omega_2^2 \neq 0 \Rightarrow$ Normal Mode: $\tilde{x}(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_2 t + C_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin\omega_2 t$

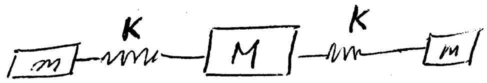
Initial displacement $\quad \tilde{x}(0) = \begin{pmatrix} x_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_1 \end{pmatrix} \Rightarrow C_1 = x_0$

$$\dot{\tilde{x}}(0) = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} \omega_2 C_2 \\ \omega_2 C_2 \end{pmatrix} \Rightarrow C_2 = \frac{V_0}{\omega_2}$$

$$\Rightarrow \tilde{x}(t) = x_0\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos\omega_0 t + \frac{V_0}{\omega_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin\omega_2 t$$

$$\xrightarrow[0]{\omega_2} \begin{pmatrix} X_0 \\ X_0 \end{pmatrix} + \begin{pmatrix} V_0 t \\ V_0 t \end{pmatrix}$$

Example 3.



guess mode shapes & natural frequencies

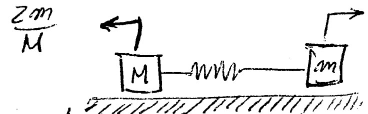
(1) $\quad \underline{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad \omega_1^2 = \frac{K}{m}$

(2) Rigid Body mode $\quad \underline{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \omega_2^2 = 0$

(3) $\quad \underline{a}_3 = \begin{pmatrix} 1 \\ -A \\ 1 \end{pmatrix}$

By Conservation of linear momentum if we subtract the rigid Body motion from the full motion, the CM should not move

$$m.1 + m.1 - MA = 0 \Rightarrow A = \frac{2m}{M} \quad \rightarrow \underline{a}_3 = \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$



the equivalent stiffness for 3rd mass

$$\tilde{K} = K.1 + K.\frac{2m}{M} = K\left(1+\frac{2m}{M}\right)$$

$\Rightarrow$ For the 1DOF oscillator $m\ddot{x}_3 + \tilde{K}x_3 = 0 \Rightarrow \omega_3^2 = \frac{\tilde{K}}{m} = \frac{K\left(1+\frac{2m}{M}\right)}{m}$

Back to $\underline{M}\ddot{\underline{x}} + \underline{K}\,\underline{x} = 0$ (*)

$\underline{K}$ pos. definite

$\Downarrow$

- $\underline{a}_1, \dots, \underline{a}_n$ mode shapes
- $\omega_1^2, \dots, \omega_n^2$ natural frequencies (squared)

General Solution (1) $\underline{X}(t) = \sum_{j=1}^{n} \left(p_j \underline{a}_j e^{i\omega_j t} + q_j \underline{a}_j e^{-i\omega_j t}\right)$

Complex Constants determined by I.C.

(2) $\Rightarrow p_j = \bar{q}_j = \gamma_j + i\delta_j$

($\underline{X}(t)$ must be real)

Substitution of (2) into (1) gives

$$\underline{X}(t) = \sum_j \left(2\gamma_j \cos\omega_j t - 2\delta_j \sin\omega_j t\right)\underline{a}_j$$

$$= \sum_{j=1}^{n} \underbrace{2\sqrt{\gamma_j^2+\delta_j^2}}_{C_j} \overbrace{\left(\underbrace{\frac{\gamma_j}{\sqrt{\gamma_j^2+\delta_j^2}}}_{\sin(\beta_j)} \cos\omega_j t - \underbrace{\frac{\delta_j}{\sqrt{\gamma_j^2+\delta_j^2}}}_{\cos(\beta_j)} \sin\omega_j t\right)}^{\sin(\omega_j t+\beta_j)} \underline{a}_j$$

$$\Rightarrow \underline{X}(t) = \sum_{j=1}^{n} C_j \underline{a}_j \sin(\omega_j t + \beta_j)$$

C_j, β_j real Constants determined by I.C.

Orthogonality of mode shapes

$$\left.\begin{array}{ll} (3) & -\omega_j^2 \underline{M}\,\underline{a}_j + \underline{K}\,\underline{a}_j = \underline{0} \\ (4) & -\omega_k^2 \underline{M}\,\underline{a}_k + \underline{K}\,\underline{a}_k = \underline{0} \end{array}\right\} \omega_j \neq \omega_k$$

$a_K^T(3) - a_j^T(4)$:

$$(\omega_2^2-\omega_1^2)\, a_K^T M a_j = 0$$

Used: $a_K^T K a_j = a_j^T K a_K$

$$a_K^T M a_j = a_j^T M a_K$$

$\Rightarrow \boxed{a_K^T M a_j = 0}$ for any $j \neq k$

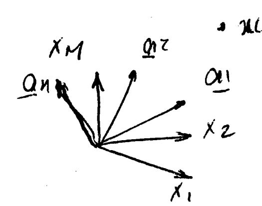
Now: $a_K^T(3) + a_j^T(4)$:

$\boxed{a_K^T K a_j = 0}$ for $j \neq K$

We can use orthogonality properties to decouple the lin. eq of motion into a system of uncoupled linear oscillations

let $\boxed{x = \Phi y}$

y modal or Principal Coordinates
projections of x onto $a_1, \dots, a_n$



$M\ddot{x} + Kx = 0$.

$$\Rightarrow M\Phi\ddot{y} + K\Phi y = 0$$

left multiply by Φ^T $\quad \Phi^T M \Phi \ddot{y} + \Phi^T K \Phi y = 0$

Note: $\Phi^T M \Phi = \begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_n^T - \end{bmatrix} \begin{bmatrix} & M & \end{bmatrix} \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ & m_2 & & \\ & 0 & \ddots & \\ & & & m_n \end{bmatrix}$

Same for K

$\Rightarrow$ (5) takes the form $\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_1 \end{pmatrix} \ddot{y} + \begin{pmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_n \end{pmatrix} y = 0$

$$\boxed{\ddot{y}_j + \underbrace{\frac{k_j}{m_j}}_{\omega_j^2} y_j = 0} \quad j = 1, \dots, n$$