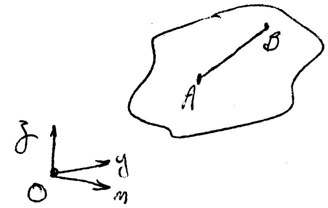
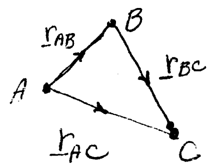
# Session 7




$$\underline{v}_B = \underline{v}_A + \underline{\omega}^A \times \underline{r}_{AB}$$

To complete the argument started last time we need to show that $\omega^A$ is in fact independent of $A$




$$\underline{V}_C = \underline{V}_A + \underline{\omega}^A \times \underline{r}_{AC}$$
$$\underline{V}_C = \underline{V}_B + \underline{\omega}^B \times \underline{r}_{BC}$$
$$\mapsto \underline{V}_A - \underline{V}_B = \underline{\omega}_B \times \underline{r}_{BC} - \underline{\omega}_A \times \underline{r}_{AC}$$

$$\underline{\omega}^A \times (-\underline{r}_{AB} + \underline{r}_{AC}) = \underline{\omega}^B \times \underline{r}_{BC}$$

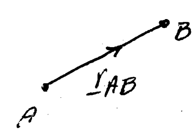
$(\underline{\omega}^A - \underline{\omega}^B) \times \underline{r}_{BC} = 0$ because $\underline{r}_{BC}$ is arbitrary

$$\Rightarrow \underline{\omega}^A = \underline{\omega}^B \overset{def}{=} \omega$$

$$\Rightarrow \underline{V}_B = \underline{V}_A + \underline{\omega} \times \underline{r}_{AB}$$

(2) Show: $\omega$ can be obtained by adding angular velocities about different axes

To see this, fix $A$ instantaneously and consider composition of $k$ rigid body rotation about $A$




$$\underline{r}_{AB}(t) = \underline{\underline{R}}(t)\, \underline{r}_{AB}(0)$$
$$= \underbrace{(\underline{\underline{R}}_k \dots \underline{\underline{R}}_2 \underline{\underline{R}}_1)}_{k \text{ rotation}} \underline{r}_{AB}(0)$$

$$\dot{r}_{AB} = (\dot{\underline{\underline{R}}}_k \underline{\underline{R}}_{k-1} \cdots \underline{\underline{R}}_1$$
$$+ \underline{\underline{R}}_k \dot{\underline{\underline{R}}}_{k-1} \cdots \underline{\underline{R}}_1$$
$$+ \underline{\underline{R}}_k \underline{\underline{R}}_{k-1} \cdots \dot{\underline{\underline{R}}}_1)\, \underline{r}_{AB}(0)$$

Recall $\dot{\underline{\underline{R}}}_i = -\underline{\underline{R}}_i \dot{\underline{\underline{R}}}_i^T \underline{\underline{R}}_i$

$$\dot{\underline{r}}_{AB} = (-\underline{\underline{R}}_k \dot{\underline{\underline{R}}}_k^T \underline{\underline{R}}_k \cdots \underline{\underline{R}}_1 + \cdots$$
$$+ \cdots$$
$$+ \underline{\underline{R}}_k \cdots \underline{\underline{R}}_2(-\underline{\underline{R}}_1 \dot{\underline{\underline{R}}}_1^T \underline{\underline{R}}_1)$$

Note: $-\underline{\underline{R}}_i \dot{\underline{\underline{R}}}_i^T \big|_{t=0} = \underline{\underline{\Omega}}_i$ , $\underline{\underline{R}}_i(0) = \underline{\underline{I}}$

$$\Rightarrow \dot{r}_{AB}(0) = (\underline{\underline{\Omega}}_K + \underline{\underline{\Omega}}_{K-1} + \cdots + \underline{\underline{\Omega}}_1)\, \underline{r}_{AB}(0)$$

$$= \sum_{i=1}^{K} \underline{\underline{\Omega}}_i\, \underline{r}_{AB}(0) = \sum_{i=1}^{K} \underline{\omega}_i \times \underline{r}_{AB}(0) = \underbrace{\left(\sum_{i=1}^{K} \omega_i\right)}_{\underline{\omega}} \times r_{AB}(0)$$

Conclusion: $\boxed{\underline{\omega} = \sum_{i=1}^{K} \omega_i}$

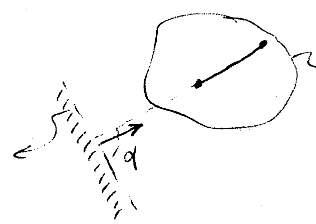
In application, how do we find $\omega$?
a) Identify all axes about which body rotates then add component angular velocities

b) If we know $\underline{v}_A$ $\underline{v}_C$ (and $\underline{r}_{AC}$) we obtain $\underline{\omega}$ by solving solution it from

$$\underline{v}_C - \underline{v}_A = \underline{\omega} \times \underline{r}_{AC}$$

c) in two-D motion (x-y) plane, $\underline{\omega} = \omega \underline{k}$

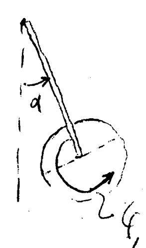
then $\omega$ can be identified as follows




Fixed direction in internal frame

then $\omega = \dot{\alpha}$

Eg.

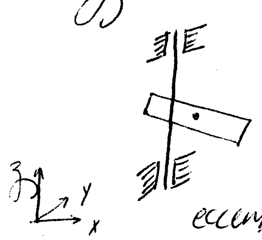



$$\underline{\omega} = (\dot{\alpha} + \dot{\varphi})\, \underline{k}$$

One last word about rigid body rotation

often some vectors are more convenient in frame that rotating with rigid body

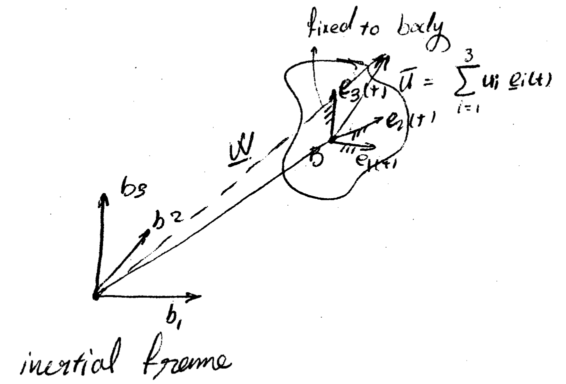
eg.




eccentric and tilted

In this example, as seen later, the angular momentum vector is easier to compute in a rotating frame $(\xi, \eta, \zeta)$

For cases like this we need to know how to evaluate the "real" (relative to inertial frame) time derivative of vectors in question




$e_i(t)$ is fixed to the body

$$\dot{W} = V_B + \dot{u}$$
$$= V_B + \sum_{i=1}^{3} [\dot{u}_i e_i(t) + u_i \dot{e}_i(t)]$$
$$= V_B + \underbrace{\sum_{i=1}^{3} \dot{u}_i e_i(t)}_{\overset{o}{u}} + \sum_{i=1}^{3} u_i \underbrace{\dot{e}_i(t)}_{\omega \times e_i}$$

$\overset{o}{u}$: relative derivative of $u$ in $[e_1, e_2, e_3]$

$\sum_{i=1}^{3} u_i \dot{e}_i(t) = \omega \times \sum_{i=1}^{3} u_i e_i = \omega \times u$

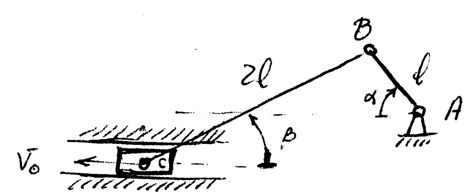
$$\dot{u} = \overset{o}{u} + \omega \times u$$

also $\dot{W} = V_B + \overset{o}{u} + \omega \times u$

## Example (2D)

$V_B = ?$

its direction is known to be perpendicular to the AB

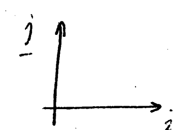



\# DOF $= 3 \times 3 - 2 - 2 - 2 - 2$

- 2 ← joint A
- 2 ← joint B → x, y at 2l rod is confined to point B of rod l
- 2 ← joint C
- 2 ← Constraint for the block

$V_B$ can be expanded in terms of say $\alpha$ and $\dot{\alpha}$ (=?) and $V_o$

$$\left.\begin{aligned} V_B &= \overset{=0}{V_A} + \omega_1 \times r_{AB} \\ V_B &= \underbrace{V_C}_{-V_o i} + \omega_2 \times r_{CB} \end{aligned}\right\} \quad \omega_1$$




$\omega_1 = -\dot{\alpha} K$

$\omega_2 = \dot{\beta} K \quad \Rightarrow \quad \hat{i}(l\dot{\alpha}\sin\alpha + V_o + 2l\dot{\beta}\sin\beta) = \hat{j}(-\dot{\alpha}l + 2l\dot{\beta}\cos\beta)$

From geometry $\quad 2l\sin\beta - a = l\sin\alpha \quad \Rightarrow \quad V_B = V_o \dfrac{\sqrt{4l^2 - (a + l\sin\alpha)^2}}{\cos\alpha\sqrt{4l^2 - (a + l\sin\alpha)^2} + a + l\sin\alpha} (\cos\alpha\,\hat{j} - \sin\alpha\,\hat{i})$