Lecture 3
The Concept of Stress, Generalized Stresses and Equilibrium

Problem 3-1: Cauchy's Stress Theorem

Cauchy's stress theorem states that in a stress tensor field there is a traction vector ***t*** that linearly depends on the outward unit normal ***n***:

$$\boldsymbol{t} = \boldsymbol{\sigma n}$$

a. Express Cauchy's stress theorem in index form.

b. Suppose the state of stress at a point in *x, y, z* coordinate system is given by the matrix below. Determine the normal stress σ_n and the shear stress τ on the surface defined by .

Problem 3-1 Solution:

(a)

$$\boxed{t_i = \sigma_{ij} n_i}$$

(b)

A **normal vector** to a plane specified by

$$f\left(x, y, z\right) = ax + by + cz + d = 0$$

is given by

$$\underline{N} = \nabla f = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where ∇f denotes the gradient.

The **unit vector** of the surface

$$2x + y - 3z = 9$$

is

$$\underline{n} = \frac{\underline{N}}{|\underline{N}|} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \frac{1}{\left|2^2 + 1^2 + (-3)^2\right|}$$

$$\underline{n} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

The magnitude of normal stress $\underline{\sigma_n}$ can be calculated by

$$\sigma_n = \underline{t} \cdot \underline{n}$$

where $\underline{t}$ is the surface traction vector

$$\underline{t} = \underline{\underline{\sigma}} \cdot \underline{n} = \begin{bmatrix} 20 & 10 & -10 \\ 10 & 30 & 0 \\ -10 & 0 & 50 \end{bmatrix} \times \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 80 \\ 50 \\ -170 \end{bmatrix}$$

Then

$$\sigma_n = \underline{t} \cdot \underline{n} = \frac{1}{14} \begin{bmatrix} 80 \\ 50 \\ -170 \end{bmatrix} \times \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \frac{720}{14}$$

$$\boxed{\sigma_n = 51\left[Force / area\right]}$$

The direction of the normal stress vector $\underline{\sigma_n}$ is $\underline{n}$

$$\underline{\sigma_n} = \sigma_n \underline{n} = \frac{51}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Shear stress vector $\underline{\tau}$ can be simply calculated by subtract the normal stress vector $\underline{\sigma_n}$ from the traction vector $\underline{t}$

$$\underline{\tau} = \underline{t} - \underline{\sigma} = \frac{1}{\sqrt{14}}\begin{bmatrix} 80 \\ 50 \\ -170 \end{bmatrix} - \frac{360}{7\sqrt{14}}\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{14}}\begin{bmatrix} -160/7 \\ -10/7 \\ -110/7 \end{bmatrix}$$

Problem 3-2: Three invariants of a stress tensor

Suppose the state of stress at a point in a *x, y, z* coordinate system is given by

$$\begin{bmatrix} 100 & 1 & 180 \\ 0 & 20 & 0 \\ 180 & 0 & 20 \end{bmatrix}$$

a. Calculate the three invariants of this stress tensor.

b. Determine the three principal stresses of this stress tensor.

Problem 3-2 Solution:

a) Invariants of stress tensor

Recall: these values do not change no matter the coordinate system selected

$$I_1 = \sigma_x + \sigma_y + \sigma_z = -100 + 20 + 20$$

$$\boxed{I_1 = -60}$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}{}^2 - \tau_{yz}{}^2 - \tau_{zx}{}^2$$
$$= (-100\times 20) + (-100\times 20) + 20^2 - 0 - 0 - 80^2$$

$$\boxed{I_2 = -10000}$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}{}^2 - \sigma_y\tau_{zx}{}^2 - \sigma_z\tau_{xy}{}^2$$
$$= (-100\times 20\times 20) + 2\times 0 - 0 - 20\times(-80)^2 - 0$$

$$\boxed{I_3 = -168000}$$

b) Principal stresses

The eigenvalue problem for a stress tensor

$$\begin{bmatrix} -100 & 0 & -80 \\ 0 & 20 & 0 \\ -80 & 0 & 20 \end{bmatrix}$$

is given by

$$\det\begin{bmatrix} -100-\lambda & 0 & -80 \\ 0 & 20-\lambda & 0 \\ -80 & 0 & 20-\lambda \end{bmatrix} = 0$$

Solve for λ , we have three eigenvalues

$$\lambda_1 = -140$$
$$\lambda_2 = 60$$
$$\lambda_3 = 20$$

The principal stresses are the three eigenvalues of the stress tensor

$$\sigma_{xp} = -140$$
$$\sigma_{yp} = 60$$
$$\sigma_{zp} = 20$$

Problem 3-4: Transformation Matrix

Suppose the state of stress at a point relative to a *x, y, z* coordinate system is given by:

$$\begin{bmatrix} 15 & -10 \\ -10 & -5 \end{bmatrix}$$

Try to find a new coordinate system *(x', y')* that corresponds to the principal directions of the stress tensor.

a. Find the principal stresses.

b. Determine the transformation matrix $[L]$.

c. Verify $[\sigma]' = [L]^T[\sigma][L]$.

Problem 3-4 Solution:

a) Determine principal stresses

The eigenvalue problem for a stress tensor

$$\begin{bmatrix} 15 & -10 \\ -10 & -5 \end{bmatrix}$$

is given by

$$\det\begin{bmatrix} 15-\lambda & -10 \\ -10 & -5-\lambda \end{bmatrix} = 0$$

Solve for λ , we have two eigenvalues

$$\lambda_1 = 19.14$$
$$\lambda_2 = -9.14$$

The principal stresses are the two eigenvalues of the stress tensor

$$\sigma_{xp} = 19.14$$
$$\sigma_{yp} = -9.14$$

b) Determine transformation matrix

Calculate the eigenvectors for the stress tensor

For $\lambda_1 = 19.14$

$$\begin{bmatrix} 15-19.14 & -10 \\ -10 & -5-19.14 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

Set $x_1 = 1$, use the first equation

$$x_2 = -0.414$$

Normalize eigenvector

$$\phi_1 = \frac{1}{\sqrt{1^2 + (-0.414)^2}} \begin{Bmatrix} 1 \\ -0.414 \end{Bmatrix}$$

$$\phi_1 = \begin{Bmatrix} 0.92 \\ -0.38 \end{Bmatrix}$$

For $\lambda_1 = -9.14$

$$\begin{bmatrix} 15-(-9.14) & -10 \\ -10 & -5-(-9.14) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

Set $x_1 = 1$, use the first equation

$$x_2 = 2.414$$

Normalize eigenvector

$$\phi_2 = \frac{1}{\sqrt{1^2 + (2.414)^2}} \begin{Bmatrix} 1 \\ 2.414 \end{Bmatrix}$$

$$\phi_1 = \begin{Bmatrix} 0.38 \\ 0.92 \end{Bmatrix}$$

Check $\phi_1 \perp \phi_2$?

$$\phi_1 \cdot \phi_2 = 0.92 \times 0.38 - 0.38 \times 0.92 = 0$$
$$\Rightarrow \phi_1 \perp \phi_2 \,!$$

Transformation Matrix

$$L = \begin{bmatrix} \phi_1 & \phi_2 \end{bmatrix}$$
$$L = \begin{bmatrix} 0.92 & 0.38 \\ -0.38 & 0.92 \end{bmatrix}$$

b) Verify

$$[\sigma]' = [L]^{\mathrm{T}} [\sigma][L]$$

If we transform the given stress tensor, will we arrive at the principal stresses?

Note: use 4 decimal places to get principal stresses

$$[L]^{\mathrm{T}} [\sigma][L] = \begin{bmatrix} 0.9238 & -0.3827 \\ 0.3827 & 0.9238 \end{bmatrix} \begin{bmatrix} 15 & -10 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} 0.9238 & 0.3827 \\ -0.3827 & 0.9238 \end{bmatrix} = \begin{bmatrix} 19.174 & 0 \\ 0 & -9.14 \end{bmatrix}$$
$$[\sigma]' = \begin{bmatrix} 19.174 & 0 \\ 0 & -9.14 \end{bmatrix}$$

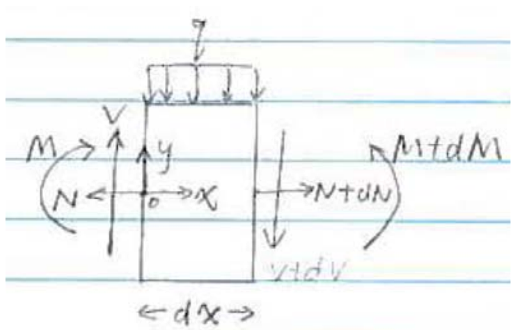
Problem 3-5: Beam Equilibrium

Derive the equation of force and moment equilibrium of a beam using the equilibrium of an infinitesimal beam element of the length dx.

Problem 3-5 Solution:

Consider an element of a beam of length dx subjected to

1. Distributed loads q
2. Shear forces V and $V+dV$
3. Moments M and $M+dM$



Equilibrium of forces in x and y direction

$$\sum F_x = 0 \Rightarrow N + dN - N = 0 \Rightarrow \boxed{\frac{dN}{dx} = 0}$$

$$\sum F_y = 0 \Rightarrow V - qdx - (V + dV) = 0 \Rightarrow \boxed{\frac{dV}{dx} = q} \cdots\cdots\cdots (*)$$

Moment equilibrium of the element at point O

$$\sum M = 0 \Rightarrow -M - qdx(\frac{dx}{2}) - (V + dV)dx + M + dM = 0$$

From equation (*),

$$\frac{dV}{dx} = -q$$

$$\Rightarrow dV = \frac{dV}{dx}dx = -qdx$$

Substitute the above equation into the moment equilibrium equation, we have

$$dM - \frac{q}{2}(dx)^2 - Vdx + q(dx)^2 = 0$$

Ignore the second-order terms, we have

$$dM = Vdx$$

$$\boxed{\frac{dM}{dx} = V}$$

To sum up, the equations of force and moment equilibrium of a beam are

$$\boxed{\begin{array}{l} \dfrac{dN}{dx} = 0 \\ \dfrac{dV}{dx} = q \\ \dfrac{dM}{dx} = V \end{array}}$$

Problem 3-6: Moderately large deflections in beams

Explain which of the three equilibrium equations below is affected by the finite rotations in the theory of the moderately large deflections of beam.

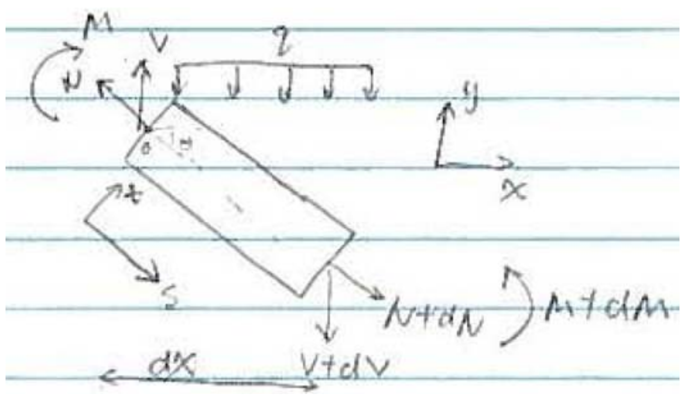
1. Moment equilibrium
2. Vertical force equilibrium
3. Axial force equilibrium

Problem 3-6 Solution:

Consider an element of a beam of length dx subjected to

1. Distributed loads q
2. Shear forces V and $V + dV$
3. Moments M and $M + dM$

and under moderately large deflection



Equilibrium of forces in t direction

$$\sum F_t = 0 \Rightarrow V\cos\theta - qdx\cos\theta - (V + dV)\cos\theta = 0$$

$$\Rightarrow \frac{dV}{dx} = -q \tag{1}$$

Note: All terms involve $\cos\theta$

Moment equilibrium of the element at point O

$$\sum M = 0 \Rightarrow -M - qdx(\frac{dx}{2}) - (V + dV)dx + M + dM = 0$$

From equation (1),

$$\frac{dV}{dx} = -q$$
$$\Rightarrow dV = \frac{dV}{dx}dx = -qdx$$

Substitute the above equation into the moment equilibrium equation, we have

$$dM - \frac{q}{2}(dx)^2 - Vdx + q(dx)^2 = 0$$

Ignore the second-order terms, we have

$$dM = Vdx$$
$$\boxed{\frac{dM}{dx} = V}$$

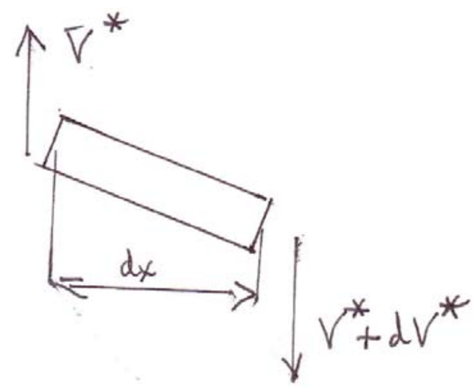
The moment equilibrium equation is not affected by moderately large deflection.

Force Equilibrium in x direction

$$\sum F_x = 0 \Rightarrow (N + dN)\cos\theta - N\cos\theta = 0$$
$$\Rightarrow \boxed{\frac{dN}{dx} = 0} \qquad (2)$$

The axial force equilibrium equation is not affected by moderately large deflection.

Define effective shear force V^* as the sum of the cross-sectional shear V and the projection of the axial force into the vertical direction



$$V^* = V + N\sin\theta = V + N\frac{dw}{dx}$$

(3)

Force equilibrium in y direction

$$\left(V^* + dV^*\right) - V^* + qdx = 0$$

$$\Rightarrow \frac{dV^*}{dx} + q = 0$$

Combining equation (3), we have

$$\frac{dV}{dx} + \frac{d}{dx}\left(N\frac{dw}{dx}\right) + q = \frac{dV}{dx} + \frac{dN}{dx}\frac{dw}{dx} + N\frac{d^2w}{dx^2} + q = 0$$

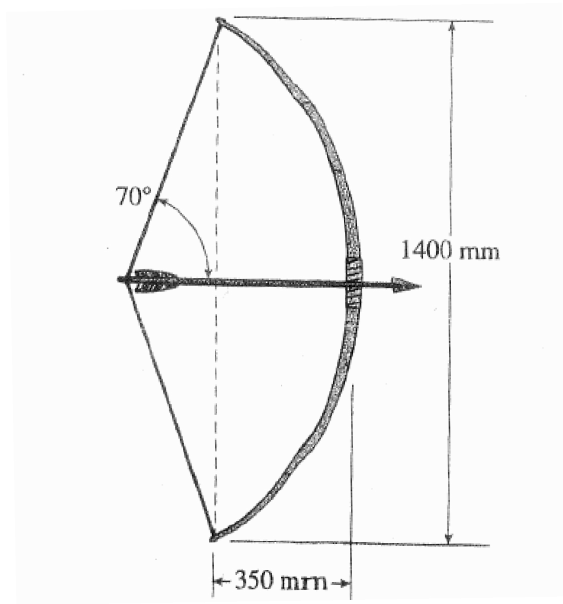
use equation(2), we have the vertical force equilibrium equation

$$\boxed{\frac{dV}{dx} + N\frac{d^2w}{dx^2} + q = 0}$$

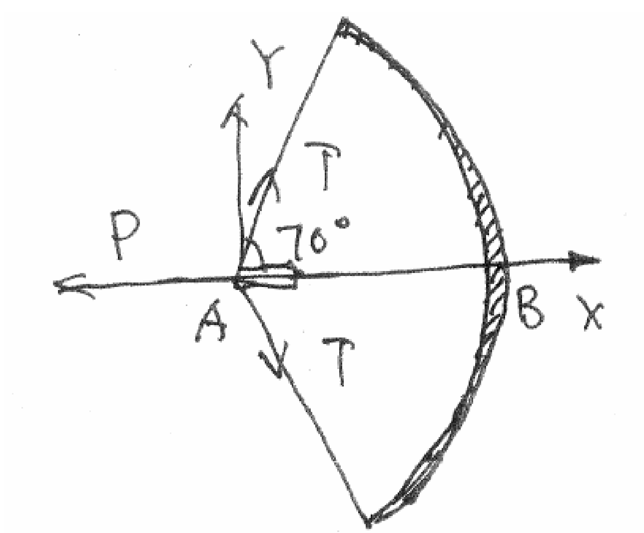
The vertical force equilibrium equation is affected by moderately large deflection.

Problem 3-7:

At full draw, and archer applies a pull of 150N to the bowstring of the bow shown in the figure. Determine the bending moment at the midpoint of the bow.



Problem 3-6 Solution:

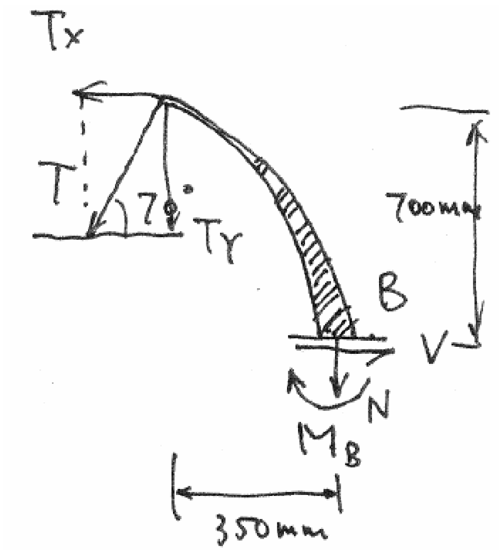


P is the pulling force at point A and T is the tension in the string. Force balance at point A

$$\sum F_x = 0$$
$$P - 2T\cos 70^{\circ} = 0$$

$$T = \frac{P}{2\cos 70^\circ} = \frac{150}{2\cos 70^\circ} = 219.3N$$

Free body diagram



where the tension T is projected in x and y direction as T_x and T_y.

Moment balance at point B

$$\sum M_B = 0$$
$$T_x \times 0.7 + T_y \times 0.35 - M_B = 0$$

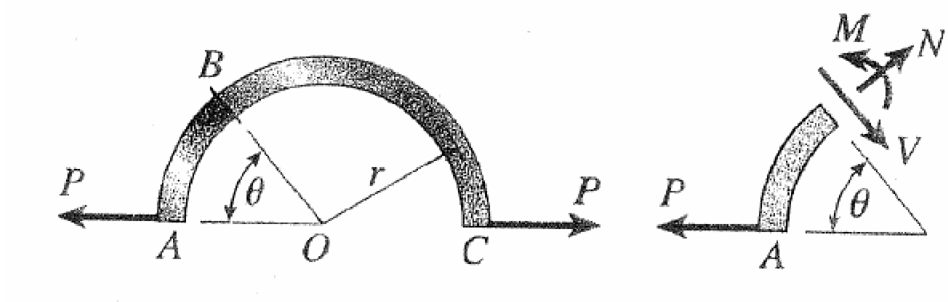
Finally, the bending moment at the midpoint of the bow is

$$\begin{aligned} M_B &= T_x \times 0.7 + T_y \times 0.35 \\ &= T \times \cos 70^\circ \times 0.7 + T \times \sin 70^\circ \times 0.35 \\ &= 219.3 \times \cos 70^\circ \times 0.7 + 219.3 \times \sin 70^\circ \times 0.35 \\ &= 124.6 N \cdot m \end{aligned}$$

Problem 3-8:

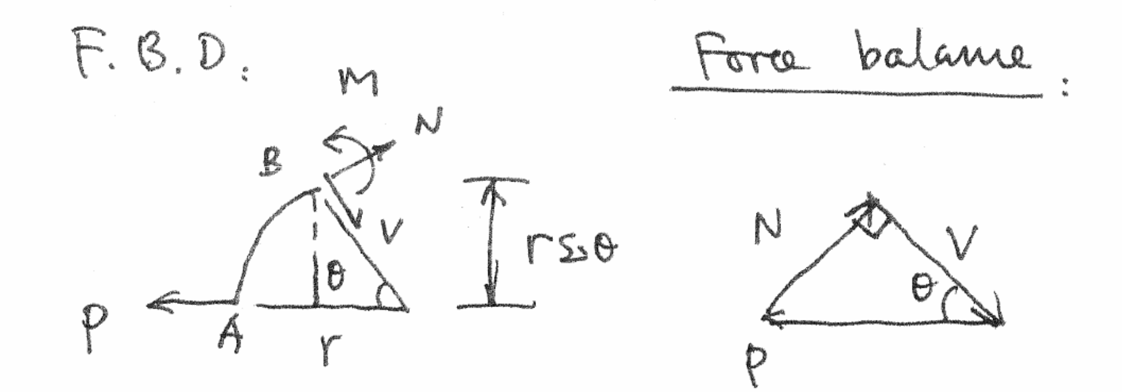
A curved bar ABC is subjected to loads in the form of two equal and opposite force P, as shown in the figure. The axis of the bar forms a semicircle of radius r.

Determine the axial force N, shear force V, and bending moment M acting at a cross section defined by the angle θ



Problem 3-8 Solution:

The free body diagram and force balance at angle θ:



We can determine axial force N, shear force V according to the force balance diagram

$$\boxed{\begin{aligned} N &= P\sin\theta \\ V &= P\cos\theta \end{aligned}}$$

Moment balance at point B

$$\sum M_B = 0$$
$$P \times r\sin\theta - M = 0$$

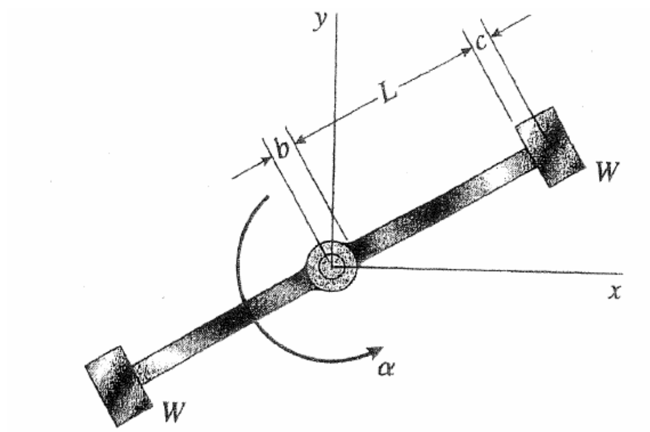
We have the bending moment M acting at a cross section defined by the angle θ

$$\boxed{M = P \times r\sin\theta}$$

Problem 3-9:

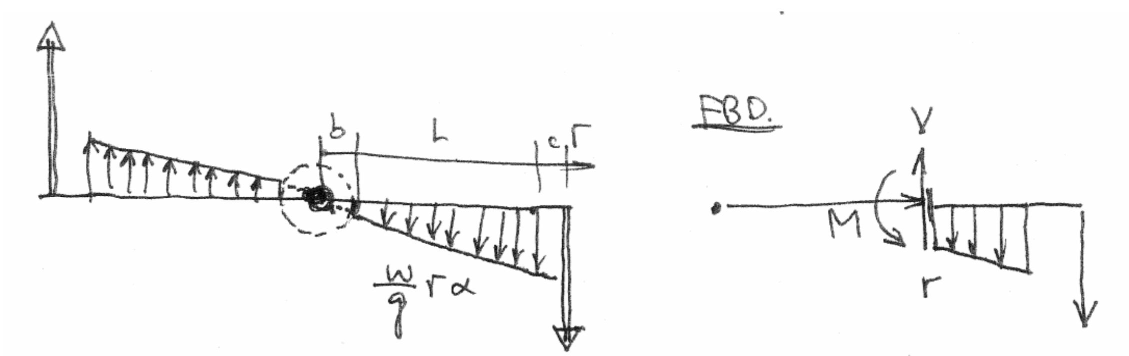
The centrifuge shown in the figure rotates in a horizontal plane (the xy plane) on a smooth surface about the z axis (which is vertical) with an angular acceleration α . Each of the two arms has weight w per unit length and supports a weight $W = 2.5wL$ at its end.

Derive formulas for the maximum shear force and maximum bending moment in the arms, assuming $b = L/9$ and $c = L/10$



Problem 3-9 Solution:

Shear force density distribution and free body diagram:



Where weight at the tip of arm W is considered as lumped mass

Force balance at any point of radius r

$$V = \int_{r}^{b+L} \frac{w}{g} r^{*} \alpha dr^{*} + \frac{W}{g}(b+L+c)\alpha$$
$$= \frac{W\alpha}{2g}\left[(b+L)^{2} - r^{2}\right] + \frac{W}{g}(b+L+c)\alpha$$

Moment balance at any point of radius r

$$M = \int_{r}^{b+L} \frac{w}{g} r^{*} \alpha \left(r^{*} - r\right) dr^{*} + \frac{W}{g}(b+L+c)\alpha(b+L+c-r)$$
$$= \frac{W\alpha}{g}\left(\frac{1}{3}r^{*3} - \frac{1}{2}rr^{*2}\right)\Bigg|_{r}^{b+L} + \frac{W\alpha}{g}(b+L+c)(b+L+c-r)$$
$$= \frac{W\alpha}{g}\left\{\frac{1}{3}\left[(b+L)^{3} - r^{3}\right] - \frac{1}{2}r\left[(b+L)^{2} - r^{2}\right]\right\} + \frac{W\alpha}{g}(b+L+c)(b+L+c-r)$$

By observation maximum shear force and bending moment occurs at r=b (closest point to the center line)

$$V_{\max} = \frac{W\alpha}{2g}\left[(b+L)^{2} - b^{2}\right] + \frac{W}{g}(b+L+c)\alpha$$
$$= \frac{W\alpha}{2g}\left[\left(\frac{L}{9}+L\right)^{2} - \left(\frac{L}{9}\right)^{2}\right] + \frac{W}{g}\left(\frac{L}{9}+L+\frac{L}{10}\right)\alpha$$

$$\boxed{V_{\max} = 3.64\frac{wL^{2}\alpha}{g}}$$

$$M_{\max} = \frac{W\alpha}{g}\left\{\frac{1}{3}\left[(b+L)^{3} - b^{3}\right] - \frac{1}{2}r\left[(b+L)^{2} - b^{2}\right]\right\} + \frac{W\alpha}{g}(b+L+c)(b+L+c-b)$$
$$= \frac{W\alpha}{g}\left\{\frac{1}{3}\left[\left(\frac{L}{9}+L\right)^{3} - \left(\frac{L}{9}\right)^{3}\right] - \frac{1}{2}r\left[\left(\frac{L}{9}+L\right)^{2} - \left(\frac{L}{9}\right)^{2}\right]\right\} + \frac{W\alpha}{g}\left(\frac{L}{9}+L+\frac{L}{10}\right)\left(L+\frac{L}{10}\right)$$

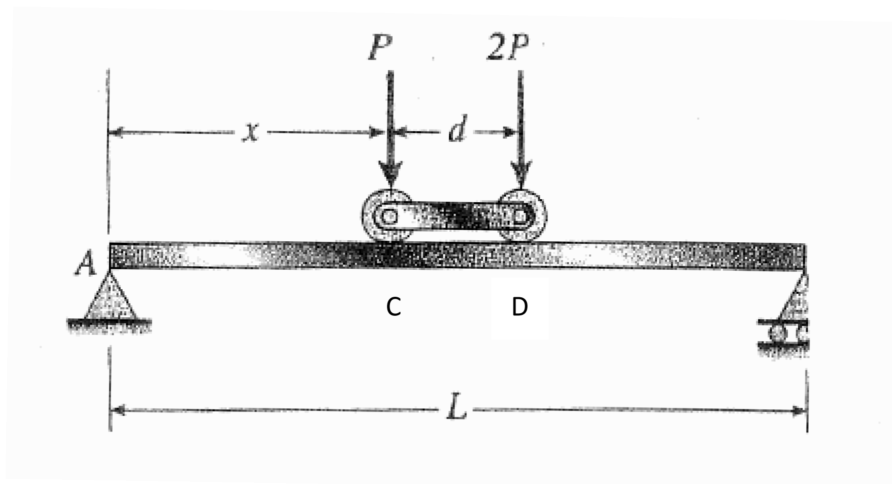
$$\boxed{M_{\max} = 3.72\frac{wL^{3}a}{g}}$$

Problem 3-10:

A simple beam AB supports two connective loads P and 2P that are distance *d* apart (see figure). The wheels may be placed at any distance x from the left support of the beam.

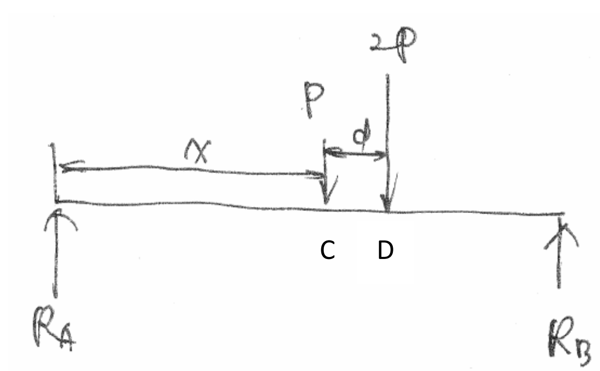
(a) Determine the distance x that will produce maximum shear force in the beam, and also determine the maximum shear force V_{max}

(b) Determine the distance that will produce the maximum bending moment, and also draw the corresponding bending moment diagram. (Assume P=10kN, d=2.4m, and L=12m)



Problem 3-10 Solution:

a) Free body diagram:



R_A R_B are reaction forces.

Moment balance at point A:

$$\sum M_A = 0$$

$$R_B \cdot L = P \cdot x + 2P(x+d)$$
$$= 3Px + 2Pd$$
$$\boxed{R_B = \frac{3Px}{L} + \frac{2Pd}{L}}$$

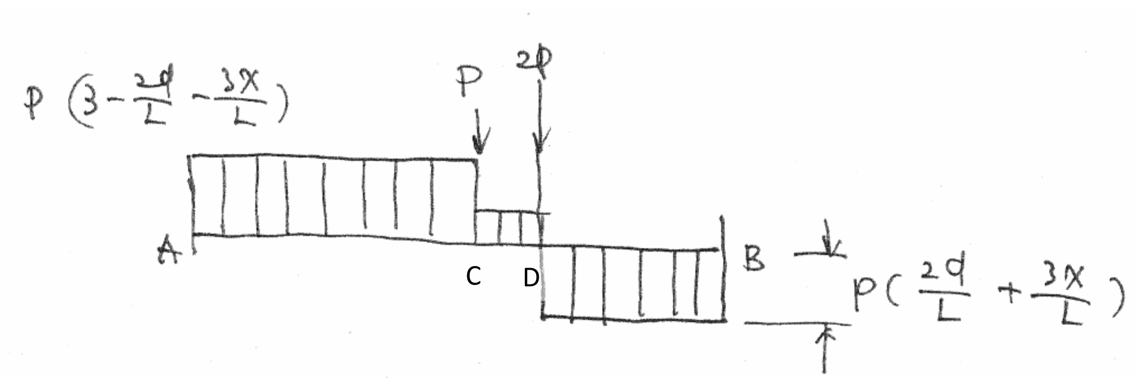
Force balance in y direction

$$\sum F_Y = 0$$

$$R_A = 3P - R_B$$

$$\boxed{R_A = P\left(2 - \frac{2d}{L} - \frac{3x}{L}\right)}$$

Shear force distribution



There are two possible maximum shear forces, $\frac{3Px}{L} + \frac{2Pd}{L}$ and $P\left(2 - \frac{2d}{L} - \frac{3x}{L}\right)$, depending on the position of the wheels x:

If $x = 0$, the two possible maximum shear forces are $V_{\max} = P \cdot \max\left\{\left(3 - \frac{2d}{L}\right), \frac{2d}{L}\right\}$

If $x = L - d$, the two possible maximum shear forces are $V_{\max} = P \cdot \max\left\{\left(3 - \frac{d}{L}\right), \frac{d}{L}\right\}$

In the range of $0 < d < L$, by observation, the maximum shear force occurs when $x = L - d$, at the left hand side of point B.

$$V_{\max} = P \cdot \left(3 - \frac{d}{L}\right) = 10 \cdot \left(3 - \frac{2.4}{12}\right) = 28kN \text{ at } x = L - d \ \left(Note: 0 < x < L - d\right)$$

b) From the shear force diagram, we know the bending moment diagram could be either Φ̇orϋ

The maximum bending moment occurs at either point C or D. Thus we need only calculate moment at these two points to determine the maximum bending moment.

Knowing $R_A = P\left(2 - \frac{2d}{L} - \frac{3x}{L}\right)$ and $R_B = \frac{3Px}{L} + \frac{2Pd}{L}$,

$$
\begin{aligned}
M_C &= R_A \cdot x = P\left[\left(3 - \frac{2d}{L}\right)x - \frac{3x^2}{L}\right] \\
M_D &= R_A\left(x+d\right) - P \cdot d \\
&= P\left[\left(3 - \frac{2d}{L}\right)x - \frac{3dx}{L} - \frac{3x^2}{L} + \left(3 - \frac{2d}{L}\right)d\right] - Pd \\
&= P\left[\left(3 - \frac{2d}{L}\right)d + \left(3 - \frac{5d}{L}\right)x - \frac{3x^2}{L}\right] - Pd
\end{aligned}
$$

Calculate maximum M_C and M_D

If $x = 0,\quad M_C = 0,\ M_D = 2Pd\left(1 - \frac{d}{L}\right)$

If $x = L - d,\quad M_C = Pd\left(1 - \frac{d}{L}\right), M_D = 0$

By observation, the maximum moment occurs at x=0, where

$$
M_D = 2Pd\left(1 - \frac{d}{L}\right) = 2 \times 10 \times 2.4 \times \left(1 - \frac{2.4}{12}\right) = 38.4 N \cdot m
$$

$$
\boxed{M_{\max} = 38.4 N \cdot m}
$$

2.080J / 1.573J Structural Mechanics
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