Lecture 9

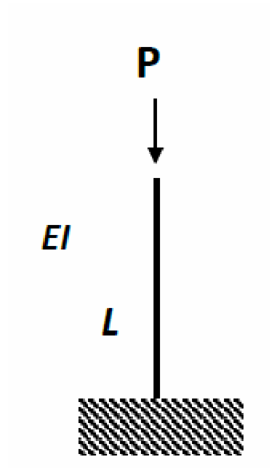
Stability of Elastic Structures

Lecture 10

Advanced Topic in Column Buckling

Problem 9-1:

A clamped-free column is loaded at its tip by a load P. The issue here is to find the critical buckling load.



a) Suggest a simple form of the buckled of the column, satisfying kinematic boundary conditions.
b) Use the Rayleigh-Ritz quotient to find the approximate value of the buckling load.
c) Come up with another buckling shape which would give you a lower value for the buckling load.
d) Find the exact solution of the problem and show the convergence of the approximate solution to the exact solution.

Follow the example of a pin-pin column, which is presented in the notes of Lecture 9.

Problem 9-1 Solution:

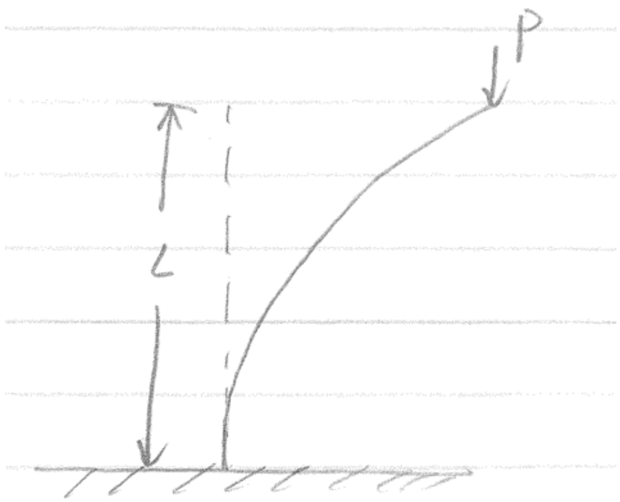
a) Kinematic boundary condition, in term of shape function $\phi(x)$, for a clamped-free column is

$$\phi(0) = \phi'(0) = 0$$

Choose a buckling shape

$$\boxed{\phi(x) = x^2}$$

$$\phi'(x) = 2x$$
$$\phi''(x) = 2$$



b) Use Rayleigh-Ritz Quotient, the critical buckling load is

$$N_c = EI \frac{\int_0^l \phi''\phi'' dx}{\int_0^l \phi'\phi' dx}$$

$$= EI \frac{\int_0^l 2\times 2 dx}{\int_0^l 2x \times 2x dx}$$

$$\boxed{N_c = 3\frac{EI}{L^2}}$$

c) Choose a buckling shape similar to a cantilever beam

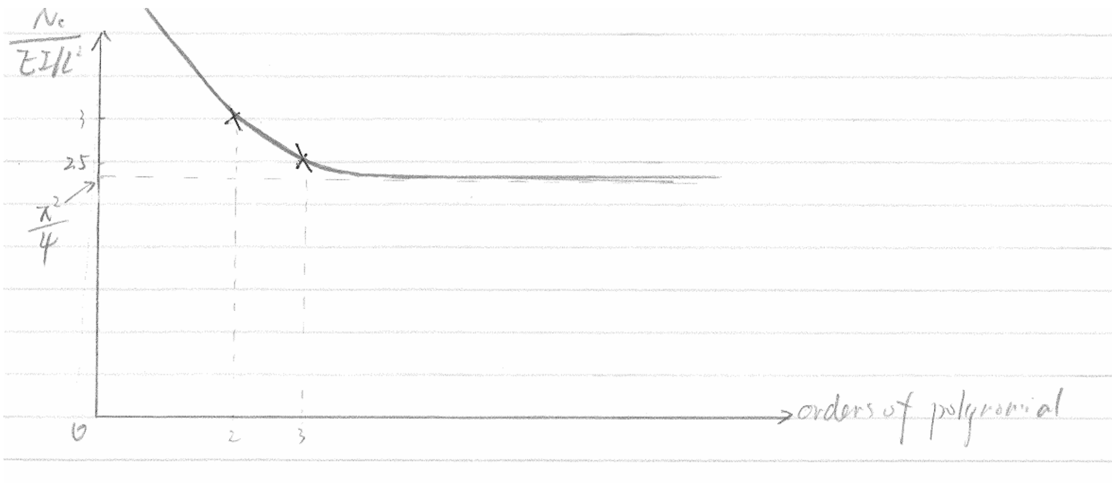
$$\boxed{\phi(x) = x^3 - 3Lx^2}$$

$$\phi'(x) = 3x^2 - 6Lx$$
$$\phi''(x) = 6x - 6L$$

$$N_c = EI \frac{\int_0^l \phi'' \phi'' \, dx}{\int_0^l \phi' \phi' \, dx}$$
$$= EI \frac{\int_0^l (6x - 6L)^2 \, dx}{\int_0^l (3x^2 - 6Lx)^2 \, dx}$$
$$= EI \frac{12L^3}{24L^5/5}$$

$$\boxed{N_c = 2.5 \frac{EI}{L^2}}$$

Compare to the result in b), $N_c = 3\frac{EI}{L^2}$, this buckling shape givers a lower value



d) Choose buckling shape

$$\boxed{\phi(x) = 1 - \cos\frac{\pi}{2L}x}$$

$$\phi'(x) = \frac{\pi}{2L}\sin\frac{\pi}{2L}x$$

$$\phi''(x) = \left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi}{2L}x$$

$$N_c = EI\frac{\int_0^l \phi''\phi''\,dx}{\int_0^l \phi'\phi'\,dx}$$

$$= EI\frac{\int_0^l \left[\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi}{2L}x\right]^2 dx}{\int_0^l \left(\frac{\pi}{2L}\sin\frac{\pi}{2L}x\right)^2 dx}$$

$$= EI\frac{\pi^2}{4L^2}$$

$$\boxed{N_c = 2.47\frac{EI}{L^2}}$$

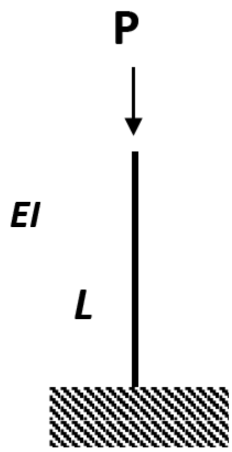
Check for local equilibrium of the solution

$$EIw^{IV} + N_c w'' = A\left[-EI\left(\frac{\pi}{2L}\right)^4 \cos\frac{\pi}{2L}x + EI\frac{\pi^2}{4L^2}\left(\frac{\pi}{2L}\right)^2 \cos\frac{\pi}{2L}x\right] = 0$$

This is the exact solution to the clamped-free buckling

Problem 9-2:

Consider a clamped-free column loaded by a compressive force at the free end.



a) Determine the critical slenderness ratio β_{crit} distinguishing between the elastic and plastic buckling response. What is the buckling stress and strain?

b) Calculate the critical plastic buckling load for $\beta = 0.5\beta_{crit}$ and the corresponding stress and strain.

c) Calculate the critical elastic buckling load for $\beta = 2\beta_{crit}$ and the corresponding stress and strain.

d) Compare all three results.

Problem 9-2 Solution:

a) First, find the bending load:

For Clamed-Free column

$$P_{cr} = \frac{\pi^2 EI}{(2L)^2} = \frac{\pi^2 EI}{4L^2}$$

Second, find the buckling stress and strain

$$\sigma_{cr}\big|_{buckling} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{4AL^2}$$

Recall that

$$r = \sqrt{\frac{I}{A}} \Rightarrow r^2 = \frac{I}{A}$$

Then

$$\sigma_{cr}\big|_{buckling} = \frac{\pi^2 E r^2}{4L^2} = \frac{\pi^2 E}{4\left(L^2/r^2\right)}$$

Recall that

$$\beta = L^2/r^2$$

$$\boxed{\begin{gathered}\sigma_{cr} = \frac{\pi^2 E r^2}{4L^2} = \frac{\pi^2 E}{4\beta^2} \\ \varepsilon_{cr} = \frac{\sigma_{cr}}{E} = \frac{\pi^2}{4\beta^2}\end{gathered}}$$

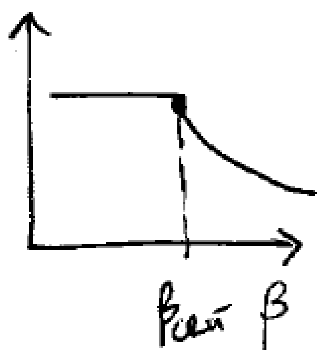
Third, find when $\sigma_{buckling} = \sigma_{yield}$

$\beta = \beta_{crit}$ when $\sigma_{cr} = \sigma_y$, which is

$$\frac{\pi^2 E}{4\beta_{crit}{}^2} = \sigma_y$$

$$\frac{\pi^2 E}{4\sigma_y} = \beta_{crit}{}^2$$

$$\boxed{\frac{\pi}{2}\sqrt{\frac{E}{\sigma_y}} = \beta_{crit}}$$



b) $\beta = 0.5\beta_{crit}$, the column yields + hits plastic buckling

$$\sigma_{cr_{pl}} = \frac{\pi^2 E_t}{4\left(\frac{1}{2}\beta_{crit}\right)^2} = \frac{\pi^2 E_t}{\beta_{crit}^{\ 2}}$$ (From Lecture Note, equation 9.73)

$$\varepsilon_{pl} = n\frac{\pi^2}{4\left(\frac{1}{2}\beta_{crit}\right)^2} = n\frac{\pi^2}{\beta_{crit}^{\ 2}}$$

c) $\beta = 2\beta_{crit}$, the column will buckle elastically

$$\sigma_{cr} = \frac{\pi^2 E}{4\beta^2} = \frac{\pi^2 E}{4\left(2\beta_{crit}\right)^2} = \frac{\pi^2 E}{16\beta_{crit}^{\ 2}}$$

$$\varepsilon_{cr} = \frac{\sigma_{cr}}{E} = \frac{\pi^2}{16\beta_{crit}^{\ 2}}$$

d) Compare the three results

	σ	ε
Yield	σ_y	ε_y
Elastic Buckling	$0.67\frac{E}{\beta_{crit}^{\ 2}} = 0.07\sigma_y$	$\frac{0.67}{\beta_{crit}^{\ 2}}$
Plastic Buckling	$\frac{19.74}{4}\frac{E}{\beta_{crit}^{\ 2}} = 0.5\sigma_y$	$\frac{7.9}{4\beta_{crit}^{\ 2}} = \frac{1.98}{\beta_{crit}^{\ 2}}$

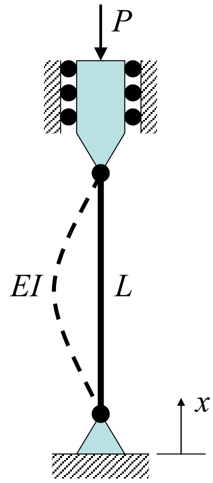
To simply our comparison, assume $n = 0.2$, $E_t = 0.5E$ (*) and recall $\pi\sqrt{\frac{E}{\sigma_y}} = \beta_{crit}$

(*)In order to compare plastic buckling to elastic+yield, we need to make future assumption about the material properties.

Problem 9-3: Consider the pin-pin column.

a) Suggest a polynomial buckling shape function $\phi(x)$ to improve the approximate solution derived in lecture note. Note that the one used in class was the parabolic shape.

b) Determine the accuracy relative to the exact solution.



Problem 9-3 Solution:

a) The exact solution is $w = \sin\left(\frac{\pi x}{L}\right)$, use the none-dimensioned value $\chi = \frac{x}{L}$, the Taylor series expansion is

$$\sin \pi\chi = \pi\chi - \frac{(\pi\chi)^3}{6} + \ldots$$

So we know the shape function must be

$$\phi(\chi) = C_1\chi + C_2\chi^3 + \ldots$$

For $0 \le x \le L/2$, the boundary conditions are

$$\begin{cases} \phi(0) = 0 \\ \phi'\left(\chi = \frac{1}{2}\right) = 0 \end{cases}$$

The first boundary condition gives

$$\phi(0) = C_1(0) + C_2(0)$$

this doesn't help.

The second boundary condition gives

$$\phi'(\chi) = C_1 + 3C_2\chi^2 = 0$$

$$\phi'\left(\chi = \frac{1}{2}\right) = C_1 + \frac{3}{4}C_2 = 0$$

$$\boxed{C_2 = -\frac{4}{3}C_1}$$

So we have

$$\phi(\chi) = C_1\chi + \left(-\frac{4}{3}C_1\right)\chi^3$$

$$\boxed{\phi(\chi) = C_1\left(\chi - \frac{4}{3}\chi^3\right)}$$

We can us the Rayleigh-Ritz Quotient

$$N_{cr} = \frac{EI\int(\phi'')^2\,dx}{\int(\phi')^2\,dx}$$

$$\phi'(x) = C_1\left(1-4\chi^2\right)d\chi/dx$$

$$\left(\phi'(\chi)\right)^2 = C_1^{\,2}\left(1-8\chi^2+16\chi^4\right)\left(d\chi/dx\right)^2$$

$$\phi''(x) = -8C_1\chi\left(d\chi/dx\right)^2$$

$$\left(\phi''(\chi)\right)^2 = 64C_1^{\,2}\chi^2\left(d\chi/dx\right)^4$$

where $d\chi/dx = \dfrac{d\left(x/l\right)}{dx} = \dfrac{1}{l}$

Since we have considered the shape function for $0 \le x \le L/2$, we must adjust the limits on the integral

$$\begin{aligned}
N_{cr} &= \frac{EI\int(\phi'')^2\,dx}{\int(\phi')^2\,dx} \\
&= EI\frac{2\int_0^{\frac{l}{2}} 64C_1^{\,2}\chi^2\left(d\chi/dx\right)^4 dx}{2\int_0^{\frac{l}{2}} C_1^{\,2}\left(1-8\chi^2+16\chi^4\right)\left(d\chi/dx\right)^2 dx} \\
&= EI\frac{\int_0^{\frac{l}{2}} 64\left(x/l\right)^2\left(1/l\right)^4 dx}{\int_0^{\frac{l}{2}}\left(1-8\left(x/l\right)^2+16\left(x/l\right)^4\right)\left(1/l\right)^2 dx} \\
&= \ldots\left(\text{after lengthly algebra}\right) \\
&= 10\frac{EI}{L^2}
\end{aligned}$$

$$\boxed{N_{cr} = 10\frac{EI}{L^2}}$$

b)

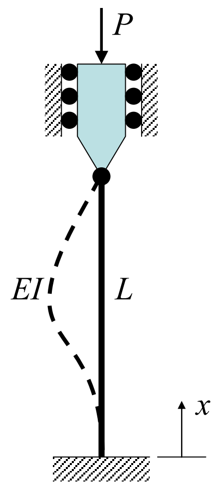
The result are compared with the polynomial used in class and the exact solution

	Exact Solution $C\sin(\pi\chi)$	Parabolic $C_1\chi + C_2\chi^2$	Cubic $C_1\chi + C_2\chi^3$
Coefficient	$\pi^2 = 9.87$	12	10
Error	N/A	21.5%	1.3%

Notice how we significantly reduce the error by including a higher order term.

Problem 9-4:

Present a step-by-step derivation of the buckling solution of the pin-clamped column from the local equilibrium equation.



Problem 9-4 Solution:

Boundary condition for this problem

$$w(0) = w(L) = 0$$
$$w'(0) = 0$$
$$EIw''(L) = 0$$

Start with 4th order ODE

$$EIw^{IV} + Pw'' = 0$$
$$w^{IV} + \frac{P}{EI} w'' = 0$$

We have an eigenvalue problem

$$\lambda^4 + \frac{P}{EI}\lambda^2 = 0$$
$$\lambda^2\left(\lambda^2 + \frac{P}{EI}\right) = 0$$
$$\lambda_1 = \lambda_2 = 0, \lambda_3 = \lambda_4 = \pm i\sqrt{\frac{P}{EI}}$$

Define $\sqrt{\frac{P}{EI}} = K \Rightarrow \lambda_3 = \lambda_4 = \pm iK$

$$\boxed{w = C_1 + C_2 x + C_3 \sin Kx + C_4 \cos Kx}$$

Use the boundary conditions to solve for constants C_1, C_2, C_3 and C_4

$w(0)= 0$

$$w(0) = 0 = C_1 + C_4$$
$$\boxed{C_1 = -C_4}$$

$w'(0) = 0$

$$w'(x) = C_2 + KC_3 \cos Kx - KC_4 \sin Kx$$
$$w'(0) = C_2 + KC_3 = 0$$
$$\boxed{C_2 = -KC_3}$$

$w(L) = 0$

$$w(L) = C_1 + C_2 L + C_3 \sin KL + C_4 \cos KL = 0$$

Substitute C_1, C_2 into the above expression

$$C_3(-KL + \sin KL) + C_4(-1 + \cos KL) = 0$$

$w''(L) = 0$

$$w''(x) = -K^2 C_3 \sin Kx - K^2 C_4 \cos Kx$$
$$w''(L) = -K^2 C_3 \sin KL - K^2 C_4 \cos KL = 0$$

$$\begin{bmatrix} -KL + \sin KL & -1 + \cos KL \\ -K^2 \sin KL & K^2 \cos KL \end{bmatrix} \begin{Bmatrix} C_3 \\ C_4 \end{Bmatrix} = 0$$

$\det[\ \ ] = 0$

$$K^2 \cos KL(-KL + \sin KL) - (-K^2 \sin KL)(-1 + \cos KL) = 0$$
$$KL \cos KL - \sin KL = 0$$
$$KL = \frac{\sin KL}{\cos KL} = \tan KL$$

So the equation to solve in order to find P_{cr} is

$$\tan KL - KL = 0$$

The smallest roots are $KL = 0$ and $KL = 4.49$,

we choose $KL = 4.49$

$$\sqrt{\frac{P}{EI}}L = 4.49$$

$$P_{cr} = \frac{20.16}{L^2}EI \approx \frac{\pi^2 EI}{\left(0.7L\right)^2}$$

$$\boxed{P_{cr} = \frac{\pi^2 EI}{\left(0.7L\right)^2}}$$

Problem 9-5:

a) Derive the solution for an imperfect clamped-free column (like that considered in problem 9-1, following a similar derivation given in the notes for a pin-pin column in the notes.

b) Find the ratio of current deflection amplitude to the amplitude of the initial imperfection such that the resulting load is 80% of the theoretical buckling load of a perfect column.

Problem 9-5 Solution:

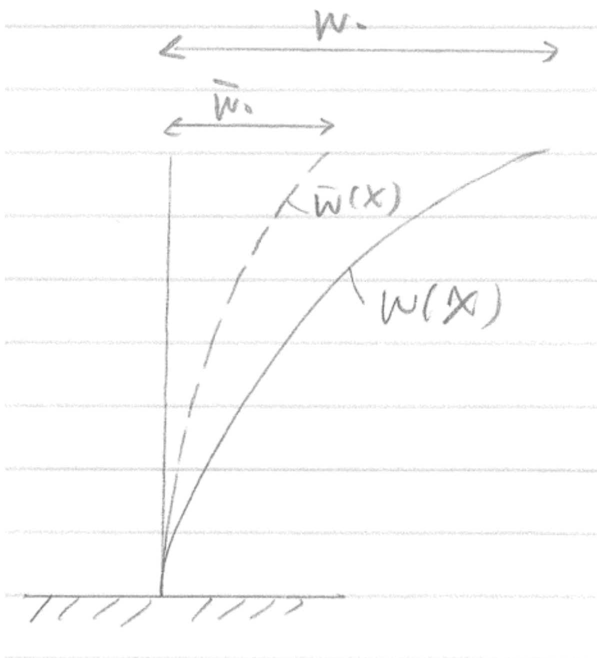
a)

$\bar{w}(x)$: shape of initial imperfection

$w(x)$: actual buckled shape

$\bar{w}_o(x)$: amplitude of initial imperfection

w_o : end amplitude of actual imperfection



Moment equilibrium of imperfect column

$$-EI\left(w-\bar{w}\right)''+P\left(w-w_o\right)=0$$

Perfect column

$$\bar{w}(x)=0$$

Assume that the initial imperfection is in the same shape as the buckling shaper

$$w(x)=w_o\left(1-\cos\lambda x\right)$$
$$\bar{w}(x)=\overline{w_o}\left(1-\cos\lambda x\right)$$

From boundary condition

$$w(L)=0$$

$$w_o \lambda^2 \cos \lambda L = 0$$

$$\lambda L = \left(\frac{2n+1}{2} \right) \pi$$

From moment equilibrium of imperfect column

$$-EI\lambda^2 \left(w - \overline{w}_o \right) \cos \lambda x + P w_o \left[1 - \left(1 - \cos \lambda x \right) \right] = 0$$

$$P w_o = EI\lambda^2 \left(w - \overline{w}_o \right)$$

Perfect column

$$\overline{w}_o = 0$$

$$P = EI\lambda^2 = \frac{\pi^2 EI}{4L^2}$$

Imperfect column

$$P w_o = P_{cr} \left(w_o - \overline{w}_o \right)$$

$$P = \frac{\pi^2 EI}{4L^2} \left(1 - \frac{\overline{w}_o}{w_o} \right)$$

$$\frac{P}{P_{cr}} = 1 - \frac{\overline{w}_o}{w_o}$$

b) When $\frac{P}{P_{cr}} = 0.8$

$$\frac{\overline{w}_o}{w_o} = 1 - \frac{P}{P_{cr}} = 0.2$$

$$\boxed{\frac{w_o}{\overline{w}_o} = 5}$$

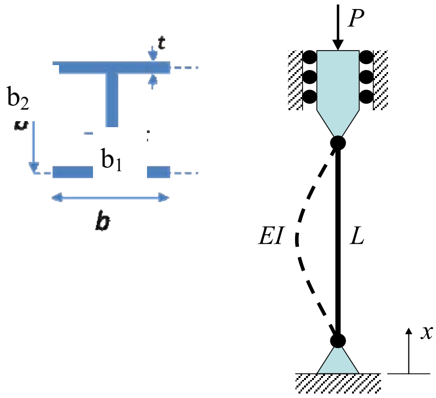
Problem 9-6:

The pin-pin elastic column of length L (shown below) is an "I" section can buckle in either plane.

a) Determine the buckling load in terms of L, b_1,b_2, t and E. Assume that $t<<b$.

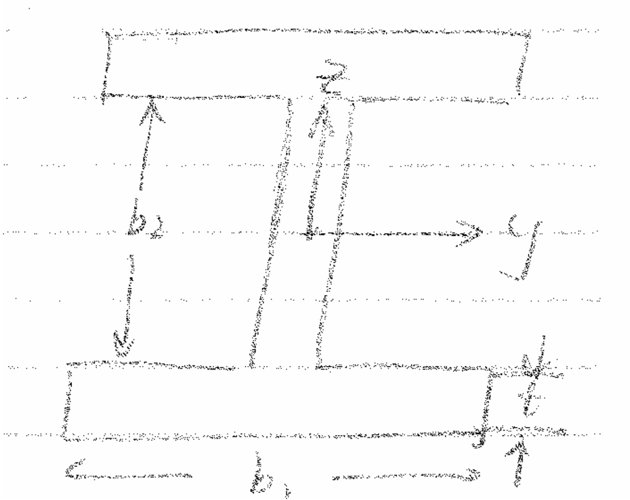
b) What should the ratio of b_1/b_2 be in order for the probability of buckling in either of the buckling planes to be the same?

Bonus: What could happen for very large width to thickness ratio?



Problem 9-6 Solution:

a) The moment s of inertia for an "I" shape cross-section is



$$I_{yy} = \frac{1}{12}tb_2^3 + 2b_1t\left(\frac{b_2}{2}\right)^2$$

$$= \frac{1}{12}tb_2^2\left(b_2 + 6b_1\right)$$

$$I_{zz} = \frac{1}{12}t^3b_2 + 2\frac{1}{12}tb_1^3$$

$$\approx \frac{1}{6}tb_1^3$$

If $I_{yy} < I_{zz}$, the column will buckle in x-z plane

$$P_{cr} = \frac{\pi^2 EI_{yy}}{l^2} = \frac{\pi^2 E}{12l^2} tb_2^{\ 2}\left(b_2 + 6b_1\right)$$

If $I_{yy} < I_{zz}$, the column will buckle in x-y plane

$$P_{cr} = \frac{\pi^2 EI_{zz}}{l^2} = \frac{\pi^2 E}{6l^2} tb_1^{\ 3}$$

b) For the probability of buckling in either of the planes to be the same , we want

$$I_{yy} = I_{zz}$$

$$\frac{1}{12} tb_2^{\ 2}\left(b_2 + 6b_1\right) = \frac{1}{6} tb_1^{\ 3}$$

$$\Rightarrow \left(\frac{b_1}{b_2}\right)^3 - 3\frac{b_1}{b_2} - \frac{1}{2} = 0$$

The only physical solution is

$$\frac{b_1}{b_2} = 1.81$$

c) If $b_1 >> t, b_2 >> t$, then local plate buckling my develop.