Random Variables

outcomes are numerical values

A hypothesis:

Birthdays are uniformly distributed over

the first six months (181 days)

and

the second six months (184 days)

of the year.

We poll n "randomly selected" people, and calculate

$$\hat{p}_{\text{first half}} = \#(\text{occurrences of Jan–June})/n$$

$$\hat{p}_{\text{second half}} = \#(\text{occurrences of July–Dec})/n$$

Should we

"<u>accept</u>" or "<u>reject</u>"

our hypothesis?

Discrete Random Variables

Introduce sample space

$\{x_1, \ldots, x_J\}$ real numbers.

Then

$\underset{\text{r.v.}}{X} = x_j$ with probability p_j, $1 \le j \le J$

where

$$\begin{cases} 0 \le p_j \le 1, \quad 1 \le j \le J \\ \displaystyle\sum_{j=1}^{J} p_j = 1 \end{cases}$$

Probability mass function (pmf):

$$\underset{\text{pmf}}{f_X}(x_j) = p_j, \quad 1 \le j \le J.$$

r.v.   outcome   probability of outcome

Note

$$\left\{ \begin{array}{l} 0 \le f_X(x_j) \le 1, \quad 1 \le j \le J \\ \text{and} \\ \displaystyle\sum_{j=1}^{J} f_X(x_j) = 1 \end{array} \right.$$

Example: uniform distribution $\quad\quad J$

Let $x_j = j, \; 1 \le j \le J$:

$J = 6$: die roll face $\quad\quad ?$

$J = 12$: birthmonth;

define

$$f_X^{\text{unif},J}(x_j) = \underbrace{1/J}_{p_j}, \quad 1 \le j \le J.$$

(Note $0 \le p_j \le 1$, $1 \le j \le J$, and $\sum_{j=1}^{J} p_j = 1$.)

Example: Bernoulli parameter θ, $0 \le \theta \le 1$

Let $J = 2$, and

$$x_1 = 0 \text{ ("tail")}, \quad x_2 = 1 \text{ ("head")}.$$

then "fair" coin: $\theta = 1/2$

$$f_X^{\text{Bernoulli}}(x;\theta) = \begin{cases} 1-\theta & \text{if } x = x_1 = 0 \quad p_1 \\ \theta & \text{if } x = x_2 = 1 \quad p_2 \end{cases}.$$

Note $0 \le p_1, p_2 \le 1$ and $p_1 + p_2 = 1$ for any admissible value of θ.

Random Variate Generation (Simulation)

X : a random variable

a sample space and probability law

x : a random variate — a realization of X

a number

Physical generation:

flip a coin, roll a die, ...

OR

Pseudo-random variate generation:

in MATLAB,

randi(J)

draws a member from the uniform pmf $f_X^{\text{unif},J}$ "population"

- a virtual roll of the die, OR
- a virtual flip of a (fair) coin, OR ...

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Expectation

Given a r.v. X with pmf $f_X(x)$, and

a univariate function g,

$$\underbrace{\mathbb{E}}_{\substack{\text{expectation}\\ \text{(not random)}}} \text{ of } \underbrace{(g(X))}_{\substack{\text{random}\\ \text{quantity}}} \equiv \sum_{j=1}^{J} \underbrace{g(x_j)}_{\text{outcome}} \cdot \underbrace{p_j}_{\text{probability } f_X(x_j)}$$

Note

$$\mathbb{E}\,(\underbrace{g(X) = C}_{\text{constant}}) = \sum_{j=1}^{J} C\, p_j = C \sum_{j=1}^{J} p_j = C.$$

μ, σ^2, and σ

mean, μ: center of mass

$$\mu = \mathbb{E}(X) = \sum_{j=1}^{J} x_j p_j$$

note $\mathbb{E}(X-\mu) = \sum_{j=1}^{J} (x_j - \mu) p_j$

$$= \sum_{j=1}^{J} x_j p_j - \sum_{j=1}^{J} \mu p_j$$

$$= \mathbb{E}(X) - \mu \sum_{j=1}^{J} p_j$$

$$= \mu \quad - \mu \quad = 0$$

variance, σ^2: "spread^2"

$$\sigma^2 \equiv \mathbb{E}\left((X-\mu)^2 \right)$$

$$= \sum_{j=1}^{J} (x_j - \mu)^2 p_j \qquad (= E(X^2) - \mu^2)$$

standard deviation, σ spread
(std dev)

$$\sigma \equiv \sqrt{\sigma^2} \qquad \text{definition}$$

Example: uniform distribution $\qquad J$

$$x_j = j, \; 1 \le j \le J \qquad p_j = 1/J, \; 1 \le j \le J$$

$$\mu = \mathbb{E}(X) = \sum_{j=1}^{J} x_j p_j = \frac{1}{J} \sum_{j=1}^{J} j = \frac{1}{J}\left(\frac{J(J+1)}{2}\right)$$

$$= \frac{J+1}{2}$$

$$\sigma^2 = \mathbb{E}((X-\mu))^2 = \frac{J^2-1}{12}$$

$$\sigma = \sqrt{\frac{J^2-1}{12}}$$

Example: Bernoulli, θ $\qquad\qquad$ $J = 2$

$$x_1 = 0, \; x_2 = 1 \qquad p_1 = 1-\theta, \; p_2 = \theta$$

$$\mu = \mathbb{E}(X) = \sum_{j=1}^{2} x_j \, p_j = 0 \cdot (1-\theta) + 1 \cdot \theta = \theta$$

$$\sigma^2 = \mathbb{E}\left((X-\mu)^2 \right) = \sum_{j=1}^{2} (x_j - \overset{\theta}{\mu})^2 \, p_j$$

$$= \theta^2 \cdot (1-\theta) + (1-\theta)^2 \, \theta = \theta \cdot (1-\theta)$$

$$\sigma = \sqrt{\theta(1-\theta)}$$

Note for $\theta \to 0$ or $\theta \to 1$, $\sigma \to 0$: "sure thing."

Functions of Random Variables

Let

given function

X distributed according to...

$$Y = g(X) \quad \text{for} \quad X \sim f_X \, .$$

new r.v.

Then for Y,

$$\text{Sample space} \equiv \{g(x_1), \ldots, g(x_{J_X})\} \text{ pruned}$$

$$\{y_1, y_2, \ldots y_{J_Y}\} \; ;$$

$$f_Y(y_i) = P(X = \text{any } x_j \text{ s.t. } g(x_j) = y_i) \qquad \cup$$

$$= \sum_{g(x_j) = y_i} f_X(x_j) \; , \quad 1 \le i \le J_Y \; .$$

Note

$$\mathbb{E}_Y(Y) = \sum_{i=1}^{J_Y} y_i \, f_Y(y_i)$$

$$= \sum_{i=1}^{J_Y} y_i \sum_{g(x_j)=y_i} f_X(x_j)$$

$$= \sum_{i=1}^{J_Y} \sum_{g(x_j)=y_i} y_i \, f_X(x_j)$$

$$= \sum_{i=1}^{J_Y} \sum_{g(x_j)=y_i} g(x_j) \, f_X(x_j) = \sum_{j=1}^{J_X} g(x_j) \, f_X(x_j)$$

each x_j appears once and only once $\qquad = E_X(g(X))\,.$

Example: uniform to Bernoulli

$$X \sim f_X^{\text{unif},J=3} \quad x_j = j,\ p_j = \frac{1}{3},\ 1 \le j \le J = 3$$

$$g(x) = \begin{cases} 0 & \text{if } x = 1 \text{ or } x = 2 \\ 1 & \text{if } x = 3 \end{cases}$$

$\Rightarrow$ $J_Y = 2$, $y_1 = 0$, $y_2 = 1$, and

$$\begin{cases} f_Y(y_1) = P(Y=0) = P(X=1 \text{ OR } X=2) \\ \qquad\qquad\qquad\qquad\quad = f_X(1) + f_X(2) = \frac{2}{3} \\ f_Y(y_2) = P(Y=1) = P(X=3) = \frac{1}{3} \end{cases}$$

Bernoulli with parameter $\theta = \frac{1}{3}$

Random Vectors

Joint pmf:

(X, Y) r. vector sample space $\{(x,y)_1, \ldots, (x,y)_J\}$

$\rightarrow \{(x_i, y_j),\ 1 \le i \le J_X,\ 1 \le j \le J_Y\}$

$$f_{X,Y}(x_i, y_j) = P(X = x_i, Y = y_j) \quad \text{AND}$$
$$= p_{i,j}^{X,Y}, \quad 1 \le i \le J_X,\ 1 \le j \le J_Y$$

where

$$\begin{cases} 0 \le p_{i,j}^{X,Y} \le 1, \quad 1 \le i \le J_X,\ 1 \le j \le J_Y \\ \displaystyle\sum_{i,j}^{J_X, J_Y} p_{i,j}^{X,Y} = 1 \end{cases}$$

Marginal pmf's

$$f_X(x_i) = P(X = x_i)$$

$$= P(X = x_i, Y = y_1 \text{ OR } X = x_i, Y = y_2 \text{ OR } \cdots)$$

$$= \sum_{j=1}^{J_Y} P(X = x_i, Y = y_j)$$

$$= \sum_{j=1}^{J_Y} f_{X,Y}(x_i, y_j) \quad , \quad 1 \le i \le J_X$$

$$f_Y(y_j) = \sum_{i=1}^{J_X} f_{X,Y}(x_i, y_j) \quad , \quad 1 \le j \le J_Y$$

Conditional pmf's

$$f_{X|Y}(x_i \mid y_j) = \frac{f_{X,Y}(x_i, y_j)}{f_Y(y_j)}$$

$1 \le i \le J_X$

$1 \le j \le J_Y$

$$f_{Y|X}(y_j \mid x_i) = \frac{f_{X,Y}(x_i, y_j)}{f_X(x_i)}$$

... Bayes' Theorem

Independence

X and Y are independent if

$$f_{X,Y}(x_i, y_j) = f_X(x_i)\, f_Y(y_j)$$

$$p^{X,Y}_{i,j} = p^X_i \cdot p^Y_j$$

$$1 \le i \le J_X$$

$$1 \le j \le J_Y$$

or

$$f_{X|Y}(x_i | y_j) = f_X(x_i)$$

$$f_{Y|X}(y_j | x_i) = f_Y(y_j) \; .$$

Expectation of sums

$$X \sim f_X\,,\quad Y \sim f_Y$$

$$\mathbb{E}_{X,Y}(g(X) + h(Y)) = \sum_{i,j} p_{i,j}^{X,Y}(g(x_i) + h(y_j))$$

$$= \sum_{i,j} p_{i,j}^{X,Y} g(x_i) + \sum_{i,j} p_{i,j}^{X,Y} h(y_j)$$

$$= \mathbb{E}_{X,Y}(g(X)) + \mathbb{E}_{X,Y}(h(Y))$$

$$\left(= \mathbb{E}_X(g(X)) + \mathbb{E}_Y(h(Y)) \quad \text{if } X, Y \text{ independent} \right)$$

Expectation of products

$X \sim f_X, \; Y \sim f_Y$ independent r.v.'s

$$\mathbb{E}(g(X)\cdot h(Y)) = \sum_{i,j} p_{i,j}^{X,Y}\, g(x_i)\, h(y_j)$$

$$= \sum_{i,j} p_i^X\, p_j^Y\, g(x_i)\, h(y_j)$$

$$= \sum_i p_i^X\, g(x_i) \sum_j p_j^Y\, h(y_j)$$

$$= \mathbb{E}_X(g(X))\, \mathbb{E}_Y(g(Y))$$

The Binomial Distribution

i.i.d. Bernoulli trials:

Let

$$X_1 \sim f_X^{\text{Bernoulli}},\ X_2 \sim f_X^{\text{Bernoulli}},\ \ldots,\ X_n \sim f_X^{\text{Bernoulli}}$$

sample from Bernoulli population for given θ

be n independent identically distributed (i.i.d.) r.v.'s.

Define new random variables

$$Z_n = \sum_{i=1}^{n} X_n \text{ (\# of 1's)}, \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_n \text{ (fraction of 1's)}.$$

sample mean

Note each experiment draws

n Bernoulli r.v.'s $\rightarrow Z_n, \bar{X}_n$.

(Pseudo) random variate generation: $\bar{X}_n$ $\theta = 1/2$

```
n = ?  % size of Bernoulli sample (r. vector)
num_exp = ?  % # of realizations of X̄_n
xbar_n_vec = zeros(1, num_exp)
for i_exp = 1 : num_exp
    bern_r_vector = randi([0,1], 1, n)
    xbar_n_vec(i_exp) = sum(bern_r_vector)/n
end
```

DEMO

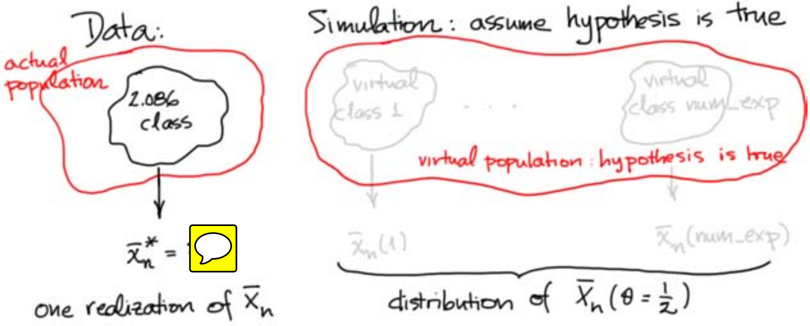
Birthmonth Revisited:

Hypothesis:

$$X = \begin{cases} 0 & \text{if birthmonth is [Jan - June]} \\ 1 & \text{if birthmonth is [July - Dec]} \end{cases}$$

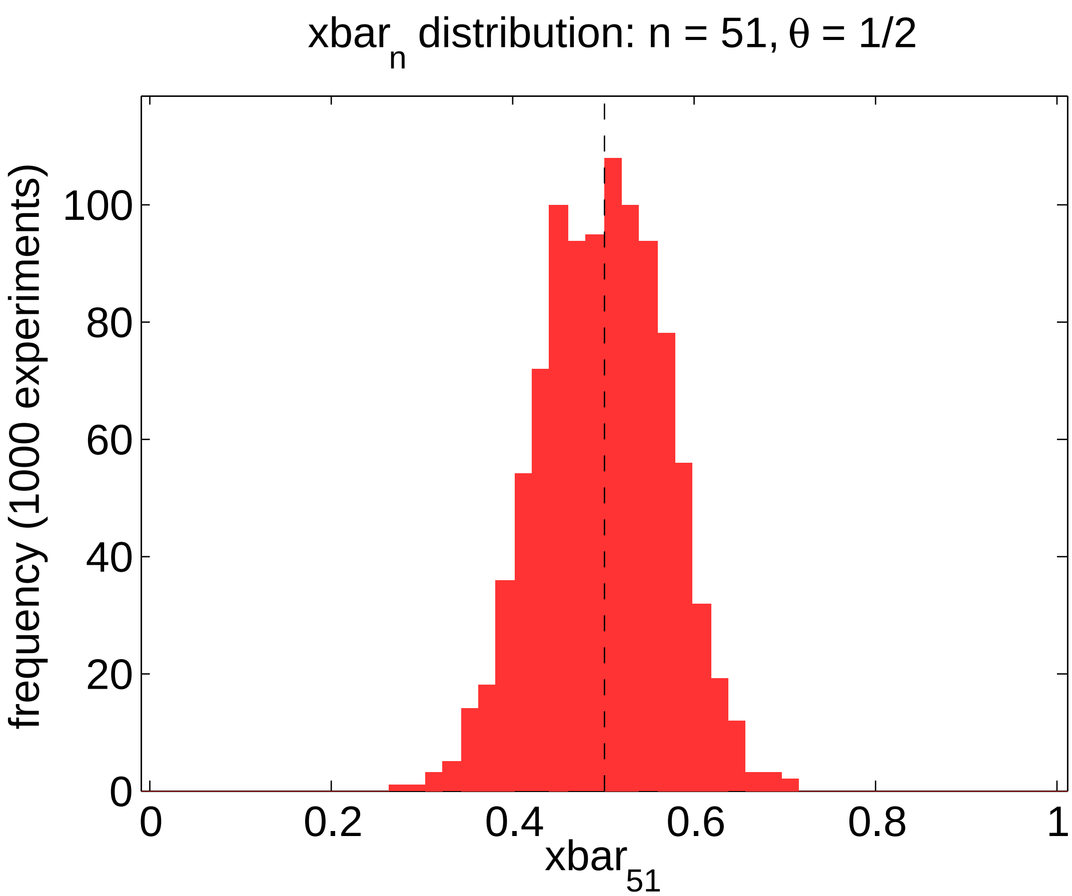
is Bernoulli with parameter $\theta = 1/2$,

$$X \sim f_X^{\text{Bernoulli}}(x; \theta = 1/2).$$



If $\bar{x}_n^*$ is extremely unlikely with respect to distribution (pmf) of $\bar{X}_n(\theta = \frac{1}{2})$

REJECT hypothesis; otherwise, ACCEPT.



xbar_n distribution: n = 51, θ = 1/2
frequency (1000 experiments)
0
20
40
60
80
100
0
0.2
0.4
0.6
0.8
1
xbar_51

properties of binomial distribution: parameter θ

pmf: or $Z_n = k$

$$P\left(\bar{X}_n = \frac{k}{n}\right) = \underbrace{\binom{n}{k} \theta^k (1-\theta)^{n-k}}_{\text{binomial}}, \quad k = 0, 1, 2, \ldots n$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

note

$$P(\bar{X}_n = 0) = 1 \cdot \theta^0 (1-\theta)^n = (\text{for } \theta = \tfrac{1}{2}) \left(\tfrac{1}{2}\right)^n$$

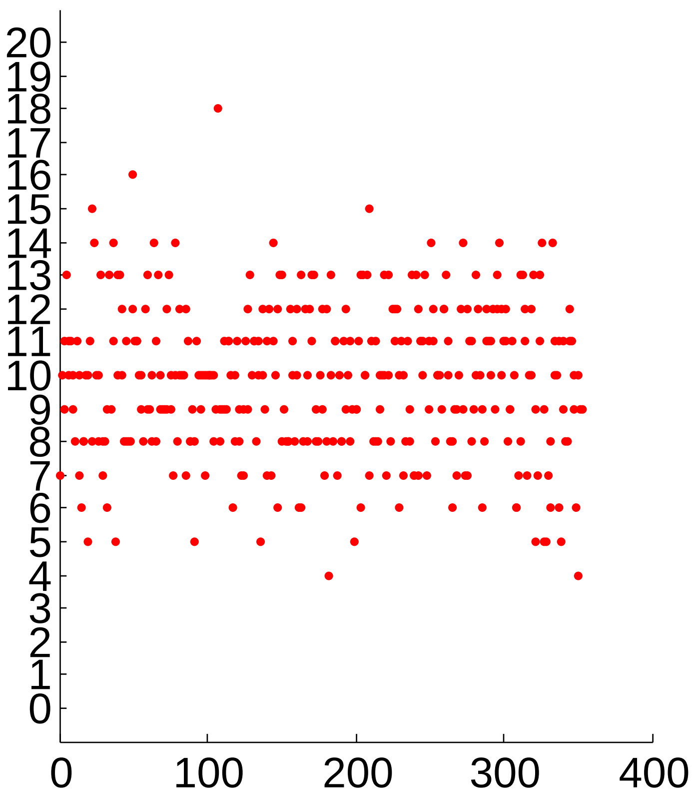
$n \to \infty$

$$\llcorner\; P(X_1 = 0 \text{ AND } X_2 = 0 \text{ AND } \ldots X_n = 0) \underset{\theta = \frac{1}{2}}{=} \tfrac{1}{2} \cdot \tfrac{1}{2} \cdots \tfrac{1}{2}$$

only one "way" to get $\bar{X}_n = 0$: $0, 0, 0, \ldots, 0$

but many ways to get $\bar{X}_n \approx \frac{1}{2}$

$0, 1, 0, 1, \ldots$ OR $1, 0, 1, 0, \ldots$ OR $1, 0, 0, 1, 1, 0, 0, 1 \ldots$ OR



20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0
0
100
200
300
400
0
0.1
0.2

mean:

$$\mathbb{E}(\bar{X}_n) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_n\right) \overset{\text{i.i.d.}}{=} \frac{1}{n}\sum_{i=1}^{n} \underbrace{\mathbb{E}_{X_n}(X_n)}_{\theta} = \theta \qquad \mu_{\bar{X}_n}$$

hence $\bar{X}_n$ is an estimator for θ

$\bar{x}_n$ is an estimate for θ

variance, std dev: Appendix A $\qquad \sigma^2_{\bar{X}_n}, \sigma_{\bar{X}_n}$

$$\mathbb{E}((\bar{X}_n - \theta)^2) = \frac{1}{n}\mathbb{E}((X - \theta)^2) = \frac{\theta(1-\theta)}{n}$$

$$\Rightarrow \sigma^2_{\bar{X}_n} = \frac{\theta(1-\theta)}{n}, \quad \sigma_{\bar{X}_n} = \sqrt{\frac{\theta(1-\theta)}{n}}$$

hence $\bar{X}_n$ is a good estimator for θ for large n,
since large deviations $|\bar{X}_n - \theta|$ are unlikely

Appendix A

$$\sigma^2_{\bar{X}_n} = \mathbb{E}\left(\left(\bar{X}_n - \theta\right)^2\right) = \mathbb{E}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \theta\right)^2\right)$$

$$= \mathbb{E}\left(\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \theta)\right)\left(\frac{1}{n}\sum_{k=1}^{n}(X_k - \theta)\right)\right)$$

$$= \frac{1}{n^2}\,\mathbb{E}\left(\sum_{i=1}^{n}\sum_{k=1}^{n}(X_i - \theta)(X_k - \theta)\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}\mathbb{E}\left((X_i - \theta)(X_k - \theta)\right)$$

but if $i \neq k$,

independence   mean

$$\mathbb{E}\left((X_i-\theta)(X_k-\theta)\right) = \mathbb{E}_{X_i}(X_i-\theta)\cdot \mathbb{E}_{X_k}(X_k-\theta) = 0$$

and hence

variance of Bernoulli r.v.

$$\sigma^2_{\bar{X}_n} = \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{E}\left((X_i-\theta)^2\right) = \frac{1}{n^2}\cdot n \cdot \theta(1-\theta)$$

$$= \frac{\theta(1-\theta)}{n}$$

$$\sigma_{\bar{X}_n} = \sqrt{\frac{\theta(1-\theta)}{n}}$$

quite famous $\sqrt{n}$

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