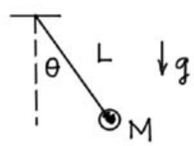
ODE IVPs

Ordinary Differential Equations
(vs PDEs)

Initial Value Problems
(vs BVPs)

Motivation

Example: Pendulum



nonlinear:

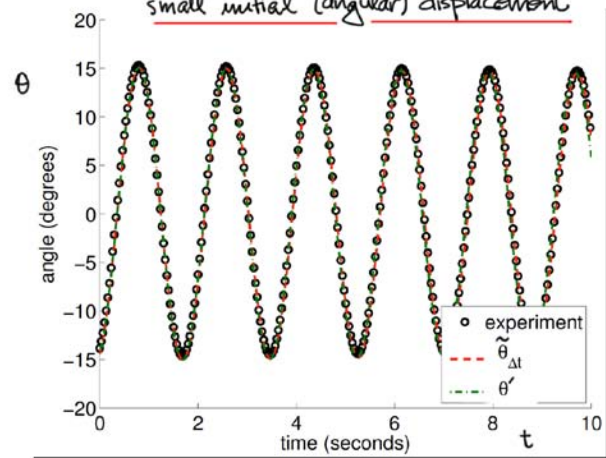
$$\begin{cases} \ddot{\theta} + (d_1\dot{\theta} + d_2|\dot{\theta}|\dot{\theta}) + \frac{g}{L}\sin\theta = 0 \\ \theta(0) = \theta_0 , \ \dot{\theta}(0) = \dot{\theta}_0 \end{cases}$$

approximation $\tilde{\theta}_{\Delta t}$

linearized: $\theta = 0 + \theta'$, $|\theta_0'| \ll 1$

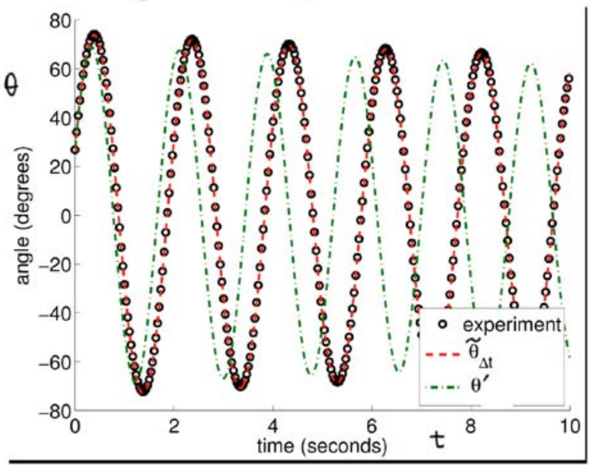
$$\begin{cases} \ddot{\theta}' + d_1\dot{\theta} + \frac{g}{L}\theta = 0 \\ \theta'(0) = \theta_0' , \ \dot{\theta}'(0) = \dot{\theta}_0' \end{cases}$$

small initial (angular) displacement



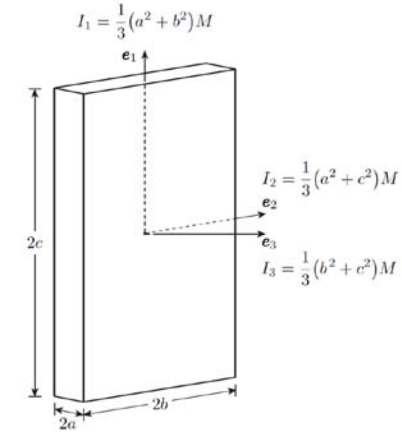
Note: d_1 (and d_2) fit to data, but
period depends only weakly on (small) damping coefficient.

large initial (angular) displacement:



MOVIES

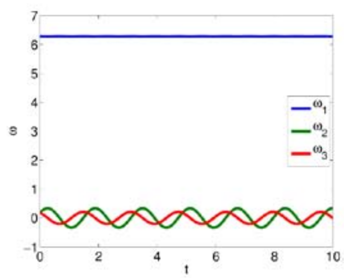
Example: Spinning Book

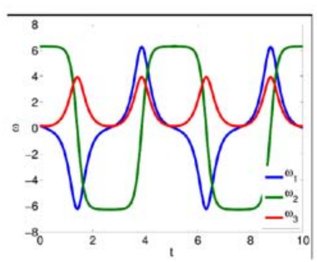


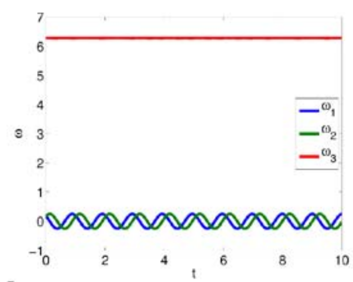
$$\omega = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$$

$$\begin{cases} I_1 \dfrac{d\omega_1}{dt} = -\omega_2\omega_3(I_3 - I_2) \\ I_2 \dfrac{d\omega_2}{dt} = -\omega_3\omega_1(I_1 - I_3) \\ I_3 \dfrac{d\omega_3}{dt} = -\omega_1\omega_2(I_2 - I_1) \end{cases}$$

$\omega_1(0)$, $\omega_2(0)$, $\omega_3(0)$ specified







$\omega(t)$ for

$\omega(0) \approx \omega_1 e_1$

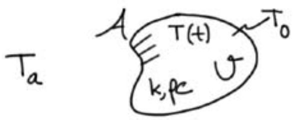
$\omega(0) \approx \omega_2 e_2$

$\omega(0) \approx \omega_3 e_3$

MOVIES

Model Problem:
Unsteady Heat Transfer

<u>Lumped Approximation</u>



assume $Bi = \dfrac{h(\mathcal{V}/A)}{k} \ll 1$

$$\underbrace{\rho c \mathcal{V} \frac{dT}{dt}}_{\substack{\text{change in}\\ \text{internal energy}}} = \underbrace{hA(T_a - T)}_{\substack{\text{heat transfer from}\\ \text{ambient to body}}} + \underbrace{\dot{q}(t)}_{\substack{\text{heat generation}\\ \text{inside body}}} \qquad W$$

$$T(t=0) = T_0$$

Let

$$\lambda = \frac{-hA}{\underbrace{\rho c \mathcal{V}}_{<0}} \qquad f(t) = \frac{\dot{q}(t)}{\rho c \mathcal{V}} \qquad \begin{aligned} u &= T - T_a \\ u_0 &= T_0 - T_a \end{aligned}$$

then

$$\begin{cases} \dfrac{du}{dt} = \lambda u + f(t), & 0 < t \leq t_f \\ u(t=0) = u_0 \end{cases}$$

<u>Solution</u>: $f(t) = 0$ $\qquad \lambda < 0$

$u(t) = u_0 e^{\lambda t}$

$$\begin{cases} \dfrac{du}{dt} = \lambda u + 0 \\ \lambda u_0 e^{\lambda t} \quad \lambda u_0 e^{\lambda t} \quad \checkmark \\ u(t=0) = u_0 \\ u_0 \quad u_0 \quad \checkmark \end{cases}$$

$$\boxed{\text{Note as } t \to \infty,\ u \to 0\ (T \to T_a).}$$

Exercise: $f(t) = 1$ $\qquad$ Exercise: Manufactured Solution

A First Numerical Scheme:
Euler Backward

Rectangle Right → Euler Backward

$$\frac{du}{dt} = \underbrace{\lambda u + f(t)}_{g(t,u(t))}, \quad 0 < t \le t_f \; ; \; u(0) = u_0$$

Assume $u(t')$, $0 < t' \le t_f$, is known: !

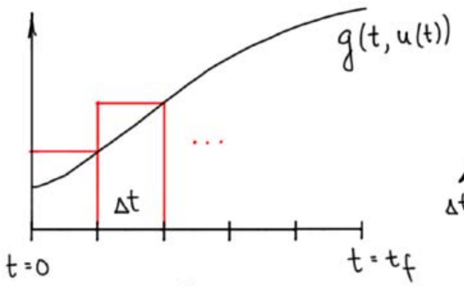
$$\int_{u_0}^{u} du = \int_0^t g(t', u(t'))dt' \qquad \text{any } t,\ 0 < t \le t_f$$

$$\Rightarrow \quad u(t) = u_0 + \int_0^t g(t', u(t'))dt'$$

e.g. $\int_0^t \lambda u_0 e^{\lambda t'} dt'$ for $f(t) = 0$

Note $g(t,w) = \lambda w + f(t)$ for given λ, $f(t)$.

Introduce a mesh and apply RR



timestep

$\Delta t = t_f / J$

$t^0 \quad t^1 \cdots t^j = j\Delta t \cdots \quad t^J \qquad 0 \le j \le J$

time level

$$\tilde{u}_{RR}(t^j) = u_0 + \Delta t \sum_{j'=1}^{j} g(t^{j'}, u(t^{j'})), \text{ OR}$$

$$\tilde{u}_{RR}(t^j) = \tilde{u}_{RR}(t^{j-1}) + g(t^j, u(t^j)), \quad 1 \le j \le J$$

$$\tilde{u}_{RR}(t^0) = u_0$$

Since

$$\tilde{u}_{RR}(t^0) = u_0$$

$$\tilde{u}_{RR}(t^1) = \underbrace{u_0}_{\tilde{u}_{RR}(t^0)} + \Delta t\, g(t^1, u(t^1))$$

$$\tilde{u}_{RR}(t^2) = \underbrace{u_0 + \Delta t\, g(t^1, u(t^1))}_{\tilde{u}_{RR}(t^1)} + \Delta t\, g(t^2, u(t^2))$$

$\vdots$

But $u(t)$ is not known:

replace $u(t^1)$ with $\tilde{u}_{RR}(t^1), \ldots$

$\Downarrow$

$$\tilde{u}(t^0) = u_0$$

$$\tilde{u}(t^1) = \tilde{u}(t^0) + \Delta t\, g(t^1, \tilde{u}(t^1))$$

$$\tilde{u}(t^2) = \tilde{u}(t^1) + \Delta t\, g(t^2, \tilde{u}(t^2))$$

$\vdots$

$$\tilde{u}(t^J) = \tilde{u}(t^{J-1}) + \Delta t\, g(t^J, \tilde{u}(t^J)) .$$

In summary, Euler Backward

$$\begin{cases} \tilde{u}(t^j) = \tilde{u}(t^{j-1}) + \Delta t\, g(t^j, \tilde{u}(t^j)), & 1 \le j \le J \\ \tilde{u}(t^0) = u_0 \end{cases}$$

or

$$\begin{cases} \tilde{u}^j = \tilde{u}^{j-1} + \Delta t\, g(t^j, \tilde{u}(t^j)), & 1 \le j \le J \\ \tilde{u}^0 = u_0 \end{cases}.$$

Note at time level j: know $\tilde{u}(t^{j-1})$;
solve for $\tilde{u}(t^j)$.

Implicit scheme: at time level j, $\tilde{u}(t^j)$ does appear in argument of $g(\,,\cdot)$.

For our model problem,

$$g(t,u) = \lambda u + f(t),$$

and hence

$$\begin{cases} \tilde{u}^0 = u_0, \\ \tilde{u}^j = \tilde{u}^{j-1} + \Delta t(\lambda \tilde{u}^j + f(t^j)), & 1 \le j \le J \end{cases}$$

or

$$\begin{cases} \tilde{u}^0 = u_0, \\ \tilde{u}^j = (\tilde{u}^{j-1} + \Delta t f(t^j))/(1 - \lambda \Delta t), & 1 \le j \le J. \end{cases}$$

for j = 1:J

2:J+1

Error Analysis: Ingredients $\lambda < 0$

equation: $du/dt = \lambda u + f(t), \; 0 < t \le t_f \;;\; u(0) = u_0$

scheme: $\begin{cases} \tilde{u}^0 = u_0 \\ \tilde{u}^j = \tilde{u}^{j-1} + \Delta t(\lambda \tilde{u}^j + f(t^j)), & 1 \le j \le J \end{cases}$

Taylor series:

$$u(t^{j-1}) = u(t^j) - \Delta t\, u_t(t^j) + \underbrace{\tfrac{\Delta t^2}{2} u_{tt}(\xi^j)}_{\equiv -\Delta t \tau^j}$$

truncation error

discretization error: $e^j \equiv u(t^j) - \tilde{u}(t^j)$

$\to 0$ as $\Delta t \to 0$

CONSISTENCY

error equation to develop bounds

$$\begin{cases} e^0 = 0 \\ (1 - \lambda \Delta t)e^j = e^{j-1} + \Delta t \tau_j, & 1 \le j \le J \end{cases}$$

Detailed derivation

$$\begin{aligned} e^j - e^{j-1} &= (u(t^j) - \tilde{u}^j) - (u(t^{j-1}) - \tilde{u}^{j-1}) \\ &= (u(t^j) - u(t^{j-1})) - (\tilde{u}^j - \tilde{u}^{j-1}) \\ &= (\Delta t\, u_t(t^j) + \Delta t \tau^j) - \Delta t(\lambda \tilde{u}^j + f(t^j)); \end{aligned}$$

$$\tilde{u}^j = (u(t^j) - e^j)$$

so

$$\begin{aligned} e^j - e^{j-1} &= \Delta t\, u_t(t^j) + \Delta t \tau^j - \Delta t \lambda u(t^j) + \Delta t \lambda e^j - \Delta t f(t^j) \\ &= \Delta t \underbrace{(u_t(t^j) - \lambda u(t^j) - f(t^j))}_{0,\text{ from equation}} + \Delta t \lambda e^j + \Delta t \tau^j \end{aligned}$$

and hence

$$\boxed{(1 - \lambda \Delta t)e^j = e^{j-1} + \Delta t \tau^j}.$$

Error Analysis: Bound $\qquad \lambda < 0$

$$\begin{cases} e^0 = 0 \\ (1-\lambda\Delta t)e^j = e^{j-1} + \Delta t\,\tau^j, & 1 \le j \le J \end{cases}$$

$$|1-\lambda\Delta t||e^j| \le |e^{j-1}| + \Delta t|\tau^j|$$

$$|e^j| \le |1-\lambda\Delta t||e^j| \text{ for all } \Delta t$$

UNCONDITIONAL STABILITY

$$\Downarrow$$

$$\begin{cases} |e^0| = 0 \\ |e^j| \le |e^{j-1}| + \Delta t|\tau^j| \end{cases}$$

Hence

$$|e^0| = 0$$

$$|e^1| \le |e^0| + \Delta t|\tau^1| = \Delta t|\tau^1|$$

$$|e^2| \le |e^1| + \Delta t|\tau^2| \le \Delta t(|\tau^1| + |\tau^2|)$$

$$|e^3| \le |e^2| + \Delta t|\tau^3| \le \Delta t \textstyle\sum_{j'=1}^{3}|\tau^{j'}|$$

$$\vdots$$

$$\boxed{|e^j| \le |e^{j-1}| + \Delta t|\tau^j| \le \Delta t \sum_{j'=1}^{j} |\tau^{j'}|}$$

$$\vdots$$

$$|e^J| \le |e^{J-1}| + \Delta t|\tau^J| \le \Delta t \sum_{j'=1}^{J} |\tau^{j'}|$$

Finally,

$$\begin{aligned} |e^j| &\le \Delta t \sum_{j'=1}^{j} |\tau^{j'}| = \Delta t \sum_{j'=1}^{j} \frac{\Delta t}{2}|u_{tt}(\xi^{j'})| \\ &\le \Delta t \sum_{j'=1}^{j} \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \\ &= \Delta t \cdot \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \sum_{j'=1}^{j} 1 \\ &= (j\Delta t)\frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \\ &= t^j \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \\ &\le t_f \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \end{aligned}$$

Note:

for $t_{\text{FIXED}} = j_{\Delta t}\Delta t \quad (j_{\Delta t} \to \infty)$,

$$\begin{aligned} |e^{j_{\Delta t}}| &= |u(j_{\Delta t}\Delta t) - \tilde{u}(t^{j_{\Delta t}})| \\ &= |u(j_{\Delta t}\Delta t) - \tilde{u}(j_{\Delta t}\Delta t)| \\ &= |u(t_{\text{FIXED}}) - \tilde{u}(t_{\text{FIXED}})| \\ &\le t_{\text{FIXED}} \cdot \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \end{aligned}$$

$$\to 0 \text{ as } \Delta t \to 0.$$

CONVERGENCE

An Explicit Scheme:
Euler Forward

Rectangle Left $\rightarrow$ Euler Forward

$$\frac{du}{dt} = \underbrace{\lambda u + f(t)}_{g(t,\,u(t))}, \quad 0 < t \le t_f\,; \quad u(0) = u_0$$

Assume $u(t')$, $0 < t' \le t_f$, is known:

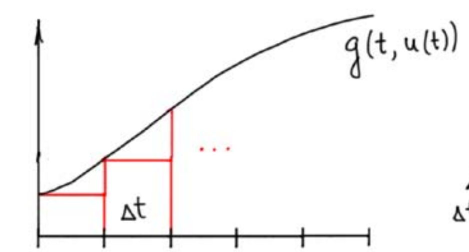
$$\int_{u_0}^{u} du = \int_0^t g(t', u(t'))\,dt' \qquad \text{any } t,\ 0 < t \le t_f$$

$$\Rightarrow \quad u(t) = u_0 + \int_0^t g(t', u(t'))\,dt'$$

e.g. $\int_0^t \lambda u_0 e^{\lambda t'} dt'$ for $f(t) = 0$

Note $g(t, w) = \lambda w + f(t)$ for given λ, $f(t)$.

Introduce a mesh and apply RL



timestep

$\Delta t = t_f / J$

$t = 0$ $\qquad$ $t = t_f$

$t^0 \quad t^1 \cdots t^j = j\Delta t \cdots \quad t^J \qquad 0 \le j \le J$

time level

$$\tilde{u}_{RL}(t^j) = u_0 + \Delta t \sum_{j'=1}^{j} g(t^{j'-1}, u(t^{j'-1})), \quad \text{OR}$$

NOTE t^{j-1}

$$\tilde{u}_{RL}(t^j) = \tilde{u}_{RL}(t^{j-1}) + g(t^{j-1}, u(t^{j-1})), \quad 1 \le j \le J$$

$$\tilde{u}_{RL}(t^0) = u_0$$

Since

$$\tilde{u}_{RL}(t^0) = u_0$$

$$\tilde{u}_{RL}(t^1) = \underbrace{u_0}_{\tilde{u}_{RL}(t^0)} + \Delta t\, g(t^0, u(t^0))$$

$$\tilde{u}_{RL}(t^2) = \underbrace{u_0 + \Delta t\, g(t^0, u(t^0))}_{\tilde{u}_{RL}(t^1)} + \Delta t\, g(t^1, u(t^1))$$

$$\vdots$$

But $u(t)$ is not known: replace $u(t^1)$ with $\tilde{u}_{RL}(t^1), \ldots$

$$\Downarrow$$

$$\tilde{u}(t^0) = u_0$$

$$\tilde{u}(t^1) = \tilde{u}(t^0) + \Delta t\, g(t^0, \tilde{u}(t^0))$$

$$\tilde{u}(t^2) = \tilde{u}(t^1) + \Delta t\, g(t^1, \tilde{u}(t^1))$$

$$\vdots$$

$$\tilde{u}(t^J) = \tilde{u}(t^{J-1}) + \Delta t\, g(t^{J-1}, \tilde{u}(t^{J-1})) \, .$$

In summary, Euler Forward

$$\begin{cases} \tilde{u}(t^j) = \tilde{u}(t^{j-1}) + \Delta t\, g(t^{j-1}, \tilde{u}(t^{j-1})), & 1 \le j \le J \\ \tilde{u}(t^0) = u_0 \end{cases}$$

or

$$\begin{cases} \tilde{u}^j = \tilde{u}^{j-1} + \Delta t\, g(t^{j-1}, \tilde{u}(t^{j-1})), & 1 \le j \le J \\ \tilde{u}^0 = u_0 \end{cases} \, .$$

Note at time level j: know $\tilde{u}(t^{j-1})$; "solve" for $\tilde{u}(t^j)$.

Explicit scheme: at time level j, $\tilde{u}(t^j)$ does NOT appear in argument of $g(\,,\cdot)$.

For our model problem,

$$g(t,u) = \lambda u + f(t) \, ,$$

and hence

$$\begin{cases} \tilde{u}^0 = u_0 \, , \\ \tilde{u}^j = \tilde{u}^{j-1} + \Delta t\left(\lambda \tilde{u}^{j-1} + f(t^{j-1})\right), & 1 \le j \le J \end{cases}$$

or

$$\begin{cases} \tilde{u}^0 = u_0 \, , \\ \tilde{u}^j = (1 + \lambda \Delta t)\tilde{u}^{j-1} + \Delta t\, f(t^{j-1}) \end{cases} \, .$$

for j = ~~1:J~~ 2:J+1

Error Analysis: Ingredients $\lambda < 0$

equation: $du/dt = \lambda u + f(t), \; 0 < t \le t_f \; ; \; u(0) = u_0$

scheme: $\begin{cases} \tilde{u}^0 = u_0 \\ \tilde{u}^j = \tilde{u}^{j-1} + \Delta t\left(\lambda \tilde{u}^{j-1} + f(t^{j-1})\right), & 1 \le j \le J \end{cases}$

Taylor series:

$$u(t^j) = u(t^{j-1}) + \Delta t\, u_t(t^{j-1}) + \frac{\Delta t^2}{2} u_{tt}(\xi^j) \quad \equiv \Delta t\, \tau^j$$

truncation error $\to 0$ as $\Delta t \to 0$ CONSISTENCY

discretization error: $e^j \equiv u(t^j) - \tilde{u}(t^j)$

error equation to develop bounds

$$\begin{cases} e^0 = 0 \\ e^j = (1 + \lambda\Delta t)\, e^{j-1} + \Delta t\, \tau_j \, , & 1 \le j \le J \end{cases}$$

Detailed derivation

$$e^j - e^{j-1} = (u(t^j) - \tilde{u}^j) - (u(t^{j-1}) - \tilde{u}^{j-1})$$

$$= (u(t^j) - u(t^{j-1})) - (\tilde{u}^j - \tilde{u}^{j-1})$$

$$= (\Delta t\, u_t(t^{j-1}) + \Delta t\, \tau^j) - \Delta t(\lambda \tilde{u}^{j-1} + f(t^{j-1}));$$

$$\tilde{u}^{j-1} = (u(t^{j-1}) - e^{j-1})$$

so

$$e^j - e^{j-1} = \Delta t\, u_t(t^{j-1}) + \Delta t\, \tau^j - \Delta t\, \lambda u(t^{j-1}) + \Delta t\, \lambda e^{j-1} - \Delta t\, f(t^{j-1})$$

$$= \Delta t \underbrace{(u_t(t^{j-1}) - \lambda u(t^{j-1}) - f(t^{j-1}))}_{0,\ \text{from equation}} + \Delta t\, \lambda e^{j-1} + \Delta t\, \tau^j$$

and hence

$$\boxed{e^j = (1 + \Delta t \lambda)\, e^{j-1} + \Delta t\, \tau^j} \ .$$

Error Analysis: Bound $\qquad \lambda < 0$

$$\begin{cases} e^0 = 0 \\ e^j = (1 + \lambda \Delta t)\, e^{j-1} + \Delta t\, \tau^j, & 1 \le j \le J \end{cases}$$

$$|e^j| \le |1 + \lambda \Delta t|\, |e^{j-1}| + \Delta t\, |\tau^j|$$

$$|1 + \lambda \Delta t|\, |e^{j-1}| \le |e^{j-1}| \text{ for all } \Delta t \le \Delta t_{cr}$$

$\Delta t_{cr} = -2/\lambda$ CONDITIONAL STABILITY

$$|1 + \lambda \Delta t| \le 1$$
$$-1 \le 1 + \lambda \Delta t \le 1$$
$$-2 \le +\lambda \Delta t \le 0$$
$$-2/\lambda \ge \Delta t$$

if $\Delta t \le \Delta t_{cr}$ $\Downarrow$

$$\begin{cases} |e^0| = 0 \\ |e^j| \le |e^{j-1}| + \Delta t\, |\tau^j| \end{cases}$$

Hence

if $\Delta t \le \Delta t_{cr}$

$$|e^0| = 0$$

$$|e^1| \le |e^0| + \Delta t\, |\tau^1| = \Delta t\, |\tau^1|$$

$$|e^2| \le |e^1| + \Delta t\, |\tau^2| \le \Delta t\, (|\tau^1| + |\tau^2|)$$

$$|e^3| \le |e^2| + \Delta t\, |\tau^3| \le \Delta t \textstyle\sum_{j'=1}^{3} |\tau^{j'}|$$

$$\vdots$$

$$\boxed{|e^j| \le |e^{j-1}| + \Delta t\, |\tau^j| \le \Delta t \sum_{j'=1}^{j} |\tau^{j'}|}$$

$$\vdots$$

$$|e^J| \le |e^{J-1}| + \Delta t\, |\tau^J| \le \Delta t \sum_{j'=1}^{J} |\tau^{j'}| \ .$$

Finally,

$$|e^j| \le \Delta t \sum_{j'=1}^{j} |\tau^{j'}| = \Delta t \sum_{j'=1}^{j} \frac{\Delta t}{2} |u_{tt}(\xi^{j'})|$$

$$\le \Delta t \sum_{j'=1}^{j} \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}|$$

if $\Delta t \le \Delta t_{cr}$

$$= \Delta t \cdot \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \sum_{j'=1}^{j} 1$$

$$= (j \Delta t)\, \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}|$$

$$= t^j\, \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}|$$

$$\le t_f\, \frac{\Delta t}{2} \max_{[0,t_f]} |u_{tt}| \ .$$

What if $\Delta t > \Delta t_{cr}$? $\quad u_0 = 1,\ f(t) = 0,\ \lambda = -1$

exact solution:

$$u_t = -u\ ,\ u(0) = 1 \Rightarrow u(t) = e^{-t}$$

Euler Forward: $\Delta t_{cr} = -2/\lambda = 2$ $\quad$ choose $\Delta t = 3$

$$\tilde{u}^0 = 1$$
$$\tilde{u}^j = \tilde{u}^{j-1} + \Delta t\,\lambda\,\tilde{u}^{j-1} = \tilde{u}^{j-1}(1+\lambda\Delta t) = -2\tilde{u}^{j-1}$$
$$\Rightarrow \tilde{u}^0 = 1,\ \tilde{u}^1 = -2,\ \tilde{u}^2 = 4,\ \ldots,\ \tilde{u}^j = (-2)^j$$
$$\Rightarrow |e^j| = |\underbrace{e^{-t^j}}_{\text{decreasing}} - \underbrace{(-2)^j}_{\text{increasing}}| > 2^j - 1 \quad (\not< \text{“bound”} = \tfrac{9}{2}j \text{ for } j \geq 5)$$

Note: blow-up even in infinite precision;
amplification of truncation error τ. $\quad$ (and...)

Note:

for $t_{FIXED} = j_{\Delta t}\Delta t \quad (j_{\Delta t} \to \infty)$,

$$|e^{j_{\Delta t}}| = |u(j_{\Delta t}\Delta t) - \tilde{u}(t^{j_{\Delta t}})|$$
$$= |u(j_{\Delta t}\Delta t) - \tilde{u}(j_{\Delta t}\Delta t)|$$
$$= |u(t_{FIXED}) - \tilde{u}(t_{FIXED})|$$
$$\leq t_{FIXED}\cdot\frac{\Delta t}{2}\max_{[0,t_f]}|u_{tt}|$$

since $\Delta t \to 0$ $\Downarrow$ $\Delta t \leq -2/\lambda$

$$\to 0 \text{ as } \Delta t \to 0.$$

CONVERGENCE, BUT: PDEs (...stiff equations);
in practice, Δt finite.

Three Basic Schemes ("θ")

EB: Euler Backward (← RR)

EF: Euler Forward (← RL)

CN: Crank-Nicolson (← trapezoidal)

Scheme	Explicit/Implicit	Stability (model problem)	Order Δt^p
EB $\tilde{u}^j = \tilde{u}^{j-1} + \Delta t\, g(t^j, \tilde{u}^j)$ solve	I	any Δt unconditional	1
EF $\tilde{u}^j = \tilde{u}^{j-1} + \Delta t\, g(t^{j-1}, \tilde{u}^{j-1})$ evaluate	E	$\Delta t \leq -2/\lambda$ conditional	1
CN $\tilde{u}^j = \tilde{u}^{j-1} + \frac{\Delta t}{2}\left(g(t^{j-1}, \tilde{u}^{j-1}) + g(t^j, \tilde{u}^j)\right)$ solve	I	any Δt unconditional (but...)	2