(GE) (BS)

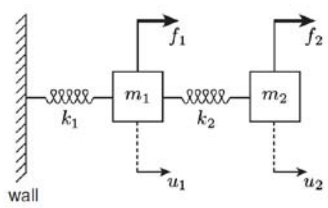
Gaussian Elimination & Back Substitution

the basic algorithm

(→ LU decomposition)

a 2×2 example

matrix equations



$k_1 = k_2 = $ "1"

$$\underset{(K)}{A}\, u = f \quad \rightarrow \quad \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

A u f

Gaussian Elimination (GE)

$$\underset{\text{pivot}}{2}\, u_1 - u_2 = f_1 \qquad \text{eqn } 1$$

$$-1 u_1 + u_2 = f_2 \qquad \text{eqn } 2$$

$m \cdot 2 - 1 = 0$

$\Rightarrow m = \frac{-\,-1}{2}$

ADD $\frac{1}{2}$ ($= \frac{-\,-1}{\text{pivot}}$) of eqn 1 to eqn 2 to obtain

$$2u_1 - u_2 = f_1 \qquad \text{eqn } 1$$

$$\underbrace{0u_1 + \frac{1}{2}u_2} = \underbrace{f_2 + \frac{1}{2}f_1} \qquad \text{eqn } 2'$$

$-1u_1 + u_2 + \frac{1}{2}(2u_1 - u_2) = f_2 + \frac{1}{2}f_1$

equal from eqn 1

equal from eqn 2

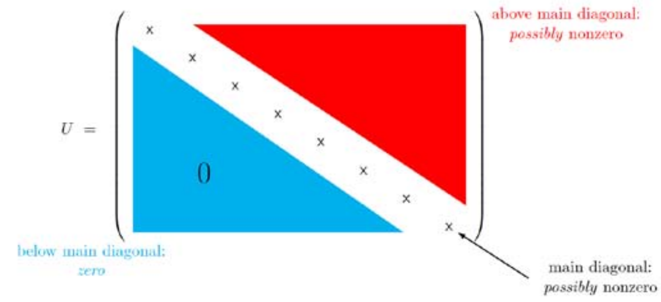
$$2u_1 - u_2 = f_1$$
$$0u_1 + \frac{1}{2}u_2 = f_2 + \frac{1}{2}f_1$$

$$\Updownarrow$$

$$\underbrace{\begin{pmatrix} 2 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} f_1 \\ f_2 + \frac{1}{2}f_1 \end{pmatrix}}_{\hat{f}}$$, or

$$Uu = \hat{f}$$

U is an upper triangular matrix



definition: U upper triangular $(n \times n)$

Back Substitution (BS)

$$\underbrace{\begin{pmatrix} 2 & -1 \\ 0 & \frac{1}{2} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} f_1 \\ f_2 + \frac{1}{2}f_1 \end{pmatrix}}_{\hat{f}} .$$

$$2u_1 - u_2 = f_1 \qquad \text{eqn 1 of } U$$
$$0u_1 + \frac{1}{2}u_2 = f_2 + \frac{1}{2}f_1 \qquad \text{eqn 2 of } U$$

Note eqn 2 of U involves only u_2, hence easy to solve:

eqn 2 of U $\qquad \frac{1}{2}u_2 = f_2 + \frac{1}{2}f_1 \;\Rightarrow\; u_2 = f_1 + 2f_2$;

and once u_2 is known, eqn 1 of U is easy to solve:

eqn 1 of U $\qquad 2u_1 - u_2 = f_1$

$$\Rightarrow 2u_1 = f_1 + \underbrace{u_2}_{\text{(already know)}}$$

$$\Rightarrow 2u_1 = f_1 + f_1 + 2f_2 = 2(f_1 + f_2)$$

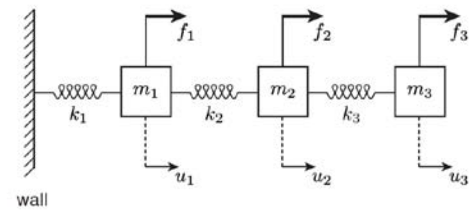
$$\Rightarrow u_1 = (f_1 + f_2) .$$

Conclusion: upper triangular system "coupled but easy."

a 3×3 example

$$\begin{cases} \text{GE:} \quad Au = f \;\Rightarrow\; Uu = \hat{f} & \text{STEP 1} \\ \text{BS:} \quad Uu = \hat{f} \;\Rightarrow\; u & \text{STEP 2} \end{cases}$$

matrix equations



$k_1 = k_2 = k_3 = 1$

$$\underset{(K)}{A}\, u = f \quad \rightarrow \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Gaussian Elimination (GE)

$$\begin{array}{ccccccc} \underset{\text{pivot}}{2} & -1 & 0 & & f_1 & \frac{1}{2}\text{ eqn } 1 & \frac{--1}{2}\cdot\text{eqn } 1 \\ -1 & 2 & -1 & & f_2 & +1\text{ eqn } 2 & \\ 0 & -1 & 1 & & f_3 & & \end{array}$$

$$\Rightarrow \quad 0u_1 + \frac{3}{2}u_2 - u_3 = f_2 + \frac{1}{2}f_1 \qquad \text{eqn } 2'$$

U-to-be

$$\tilde{U}(k=1) \equiv \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \tilde{\hat{f}}(k=1) \equiv \begin{pmatrix} f_1 \\ f_2 + \frac{1}{2}f_1 \\ f_3 \end{pmatrix}$$

"free"

$\hat{f}$-to-be

after first column is zeros below main diagonal

$$\tilde{U}(k=1)\, u = \tilde{\hat{f}}(k=1)$$

$$\begin{array}{cccccc} \multicolumn{3}{c}{\overbrace{\hspace{6em}}^{\tilde{U}(k=1)}} & \tilde{\hat{f}}(k=1) & & \\ 2 & -1 & 0 & f_1 & & \\ 0 & \underset{\text{pivot}}{\frac{3}{2}} & -1 & f_2 + \frac{1}{2}f_1 & \frac{2}{3}\text{ eqn } 2' & \frac{--1}{3/2} \\ 0 & -1 & 1 & f_3 & 1\text{ eqn } 3' & \end{array}$$

$$\Rightarrow \quad \begin{array}{ccccc} 2 & -1 & 0 & & f_1 \\ 0 & \frac{3}{2} & -1 & & f_2 + \frac{1}{2}f_1 \\ 0 & 0 & \frac{1}{3} & & f_3 + \frac{2}{3}f_2 + \frac{1}{3}f_1 \end{array} \quad , \quad \text{or}$$

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}}_{u} \underset{=}{=} \underbrace{\begin{pmatrix} f_1 \\ f_2 + \frac{1}{2}f_1 \\ f_3 + \frac{2}{3}f_2 + \frac{1}{3}f_1 \end{pmatrix}}_{\hat{f}}$$

$\tilde{U}(k=n-1)$ $\qquad$ $\tilde{\hat{f}}(k=n-1)$

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Back Substitution (BS)

eqn $n(=3)$ of U $\quad \frac{1}{3}u_3 = f_3 + \frac{2}{3}f_2 + \frac{1}{3}f_1 \quad \Rightarrow \quad u_3 = 3f_3 + 2f_2 + f_1.$

eqn 2 of U

$$\frac{3}{2}u_2 - \underbrace{u_3}_{\substack{\text{known;} \\ \text{(move to r.h.s.)}}} = f_2 + \frac{1}{2}f_1$$

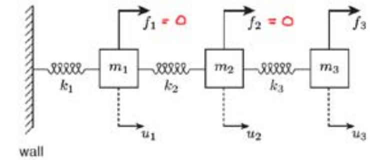
$$\frac{3}{2}u_2 = f_2 + \frac{1}{2}f_1 + u_3 \quad \Rightarrow \quad u_2 = 2f_2 + f_1 + 2f_3.$$

eqn 1 of U

$$2u_1 - \underbrace{u_2}_{\substack{\text{known;} \\ \text{(move to r.h.s.)}}} + \underbrace{0 \cdot u_3}_{\substack{\text{known;} \\ \text{(move to r.h.s.)}}} = f_1$$

$$2u_1 = f_1 + u_2 \; (+\, 0 \cdot u_3) \quad \Rightarrow \quad u_1 = f_1 + f_2 + f_3.$$

a physical interpretation



$k_1 = k_2 = k_3 = 1$

$$\begin{array}{ccc} \begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} & \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = & \begin{pmatrix} f_1 \\ f_2 + \frac{1}{2}f_1 \\ f_3 + \frac{2}{3}f_2 + \frac{1}{3}f_1 \end{pmatrix} \begin{array}{c} 0 \\ 0 \\ f_3 \end{array} \\ U & u \quad = & \hat{f} \end{array}$$

$U_{33}\, u_3 = f_3 \quad \Rightarrow \quad k_{eff} = U_{33} \; (= 1/3)$

springs in series

the General Case: $n \times n$

review of procedure

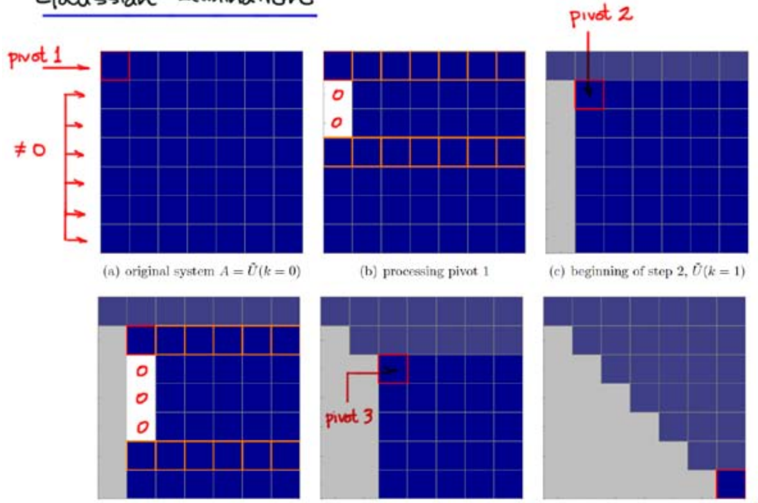
STEP 1: $\underset{n\times n}{A} \; \underset{n\times 1}{u} = \underset{n\times 1}{f} \quad \rightarrow \quad \underset{n\times n}{U} \; \underset{n\times 1}{u} = \underset{n\times 1}{\hat{f}}$

Gaussian Elimination (GE)

STEP 2: $Uu = \hat{f} \quad \Rightarrow \quad u$

Back Substitution (BS)

Gaussian Elimination



(a) original system $A = \tilde{U}(k=0)$ (b) processing pivot 1 (c) beginning of step 2, $\tilde{U}(k=1)$

(d) processing pivot 2 (e) beginning of step 3, $\tilde{U}(k=2)$ (f) final matrix $U = \tilde{U}(k=n-1)$

FLOPs to form U: $n \to \infty$

to form $\tilde{U}(k=1)$:

eqn 2 $\to$ eqn 2′ : $1 + 2\cdot(n-1) = 2n-1$

eqn 3 $\to$ eqn 3′ : $= 2n-1$

$\vdots$

eqn n $\to$ eqn n' : $= 2n-1$

$\overline{\qquad\qquad}$ $(2n-1)\cdot(n-1)$

to form $\tilde{U}(k=2)$:

eqn 2′ $\to$ eqn 2″ : $2n-3$

$\vdots$

$\overline{\qquad\qquad}$ $(2n-3)\cdot(n-2)$

$\vdots$

to form $\tilde{U}(k=n-1)$: 3

$$= (2n-1)\cdot(n-1) + (2n-3)\cdot(n-2) + (2n-5)\cdot(n-3) + \cdots 3$$

$$= \sum_{k=1}^{n-1} (2(n-k)+1)\cdot(n-k)$$

$$\sim \sum_{k=1}^{n-1} 2(n-k)^2 = 2\sum_{q=1}^{n-1} q^2 \qquad \boxed{q = n-k}$$

But,

$$2\sum_{q=1}^{n-1} q^2 = 2(n-1)^2(n-1)\underbrace{\sum_{q=1}^{n-1}\left(\frac{q}{n-1}\right)^2\frac{1}{n-1}}\ ;$$

recall "rectangle rule, right":

$$\int_0^1 x^2\,dx \text{ on } n-1 \text{ segments } (h = 1/n-1) \cong \overbrace{\sum_{q=1}^{n-1}\underbrace{\left(q\cdot\frac{1}{n-1}\right)^2}_{x^2}\underbrace{\left(\frac{1}{n-1}\right)}_{h}}$$

$$\left(\begin{matrix} n-1 = 2 \\ (h = 1/2) \end{matrix} \quad \underset{x=0 \quad h \quad 2h=1}{\vdash\!\!\vdash\!\!\dashv} : \ \underbrace{h^2}_{(\frac{1}{2})^2}\cdot h + \underbrace{(2h)^2}_{1^2}\cdot h \right)$$

$$\xrightarrow[\substack{h\to 0,\\ \text{or } n\to\infty}]{} \int_0^1 x^2\,dx = 1/3\ ;$$

hence

FLOPs to form $U \sim 2(n-1)^3 \cdot \frac{1}{3}$

$$\boxed{\sim \tfrac{2}{3} n^3} \quad \text{as } n \to \infty .$$

Similarly,

FLOPs to form $\hat{f} \sim \boxed{n^2}$ as $n \to \infty$.

Thus, total cost of GE ($\Rightarrow U, \hat{f}$) is

$$\boxed{\tfrac{2}{3} n^3 \quad \text{as } n \to \infty} \quad .$$

Back Substitution (BS)

$$\begin{pmatrix} U_{11} & U_{12} & \cdots & \cdots & U_{1n} \\ & U_{22} & & & U_{2n} \\ & & \ddots & & \vdots \\ & 0 & & U_{n-1\,n-1} & U_{n-1\,n} \\ & & & & U_{n\,n} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{n-1} \\ \hat{f}_n \end{pmatrix}$$

$$U \qquad u = \hat{f}$$

		FLOPs
eqn n:	$U_{n\,n} u_n - \hat{f}_n \Rightarrow u_n = \dfrac{\hat{f}_n}{U_{n\,n}}$	1
eqn $n-1$:	$U_{n-1\,n-1} u_{n-1} + U_{n-1\,n} u_n = \hat{f}_{n-1}$	
	$\Downarrow$	
	$U_{n-1\,n-1} u_{n-1} = \hat{f}_{n-1} - U_{n-1\,n-1} u_{n-1} \Rightarrow u_{n-1}$	3
$\vdots$		$\vdots$
eqn 1:	$U_{11} u_1 + U_{12} u_2 + \cdots + U_{1\,n} u_n = \hat{f}_1$	
	$\Downarrow$	
	$U_{11} u_1 = \hat{f}_1 - U_{12} u_2 - \cdots - U_{1\,n} u_n \Rightarrow u_1 .$	$2n-1$

$$= \sum_{k=1}^{n} 2k - 1 \sim \boxed{n^2 \text{ as } n \to \infty} \quad \ll \text{cost of GE } (\tfrac{2}{3} n^3)$$

summary of operation counts

full, or "dense", matrices

inner product w, v $n \times 1$

$$w^T v = \sum_{i=1}^{n} w_i v_i = w_1 v_1 + w_2 v_2 + \cdots + w_n v_n$$

$$\sim \boxed{2n} \text{ FLOPs as } n \to \infty$$

matrix-vector product A $n \times n$, w, v $n \times 1$

$$w = Av \qquad w_i = \underbrace{\sum_{j=1}^{n} A_{ij} v_j}_{\text{i.p.: } 2n \text{ FLOPs}} , \quad 1 \le i \le n$$

$$\cdot \; n \sim \boxed{2n^2} \text{ FLOPs as } n \to \infty$$

GE (+BS)

$$Au = f \qquad \boxed{\tfrac{2}{3} n^3} \quad \text{as } n \to \infty$$

also matrix-matrix product
($\equiv$ n matrix-vector products)

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