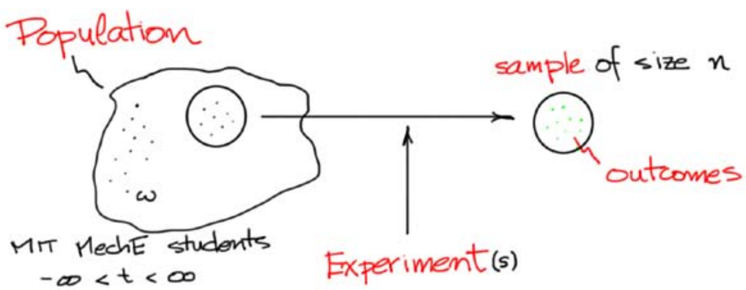
Probability (& Statistics)

Basic Concepts
(Frequentist View)

A "Calculus" for Frequency



Experiment: Two "Variables"

1) Heat Transfer <u>Background</u>: Coursework
   N(ot taken yet) or T(aken) 2.005

2) Heat Transfer <u>Knowledge</u>: "Wall" Question
   W(rong) or R(ight) answer

Experiment : $\omega \rightarrow \underbrace{\{\overset{O_1}{(N,W)};\ \overset{O_2}{(N,R)};\ \overset{O_3}{(T,W)};\ \overset{O_4}{(T,R)}\}}_{\text{sample space of all possible outcomes}}$

Each experiment yields one, and only one, outcome:

	(N,W)	Not taken 2.005 AND Wrong answer;	O_1
OR	(N,R)	Not taken 2.005 AND Right answer;	O_2
OR	(T,W)	Taken 2.005 AND Wrong answer;	O_3
OR	(T,R)	Taken 2.005 AND Right answer.	O_4

AND

"Number" function: for an event $\mathcal{E}(N,T,W,R)$,

$$\#(\mathcal{E}) \equiv \text{number of members of sample for which outcome satisfies } \mathcal{E}.$$

ex: if $\mathcal{E} \equiv N$ (ot taken), $\#(\mathcal{E}) \equiv$ number of members of sample for which outcome is (N,W) OR (N,R)

"Frequency" function:

$$\varphi_n(\mathcal{E}) = \#(\mathcal{E})/\underbrace{n}_{\text{sample size}} \quad \text{(fraction of occurrences)}.$$

"Joint" frequencies: $\{(N,W); (N,R); (T,W); (T,R)\}$

$$\underbrace{\varphi_n((N,W))}_{\text{fraction of outcomes which are } (N,W)},\ \varphi_n((N,R)),\ \varphi_n((T,W)),\ \varphi_n((T,R)).$$

Note:

$$\varphi_n((N,W)) + \varphi_n((N,R)) + \varphi_n((T,W)) + \varphi_n((T,R))$$

$$= \left(\#((N,W)) + \#((N,R)) + \#((T,W)) + \#((T,R))\right)/n$$

$$= 1$$

since each experiment yields one and only one outcome and hence is counted in one and only one of $\#((N,W)), \#((N,R)), \#((T,W)), \#((T,R))$.

"Marginal" Frequencies: (Heat Transfer) Background

Interpret Background as a "variable" which can take on two values: N or T.

Note:

an experiment can not yield N $\overset{\cap}{\text{AND}}$ T:
events N and T are mutually exclusive;

an experiment must yield either N $\overset{\cup}{\text{OR}}$ T:
events N and T are collectively exhaustive.

Sample

Hence

$$\#(N \overset{\cap}{\text{AND}} T) = 0$$

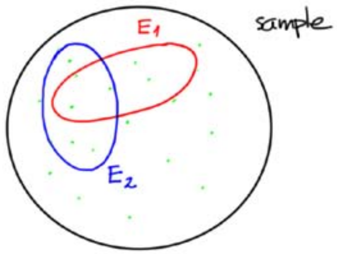
$$\Rightarrow \varphi_n(N \text{ AND } T) = 0 \qquad \text{mutually exclusive}$$

and

$$\#(N \overset{\cup}{\text{OR}} T) = n$$

$$\Rightarrow \varphi(N \text{ OR } T) = 1. \qquad \text{collectively exhaustive}$$

Generally,



$$\#(E_1 \overset{\cup}{\text{OR}} E_2) = \#(E_1) + \#(E_2) - \#(E_1 \overset{\cap}{\text{AND}} E_2)$$

$$\Downarrow$$

$$\varphi_n(E_1 \text{ OR } E_2) = \varphi_n(E_1) + \varphi_n(E_2) - \underbrace{\varphi_n(E_1 \text{ AND } E_2)}_{\substack{\text{zero if} \\ \text{mutually exclusive}}} .$$

Here,

$$\mathcal{E} \equiv N \equiv (N,W) \text{ OR } (N,R),$$

satisfied iff

so

$$\varphi_n(N) = \varphi_n((N,W)) + \varphi_n((N,R))$$

$$- \underbrace{\varphi_n((N,W) \text{ AND } (N,R))}_{\substack{\text{zero: outcomes of sample space are} \\ \text{mutually exclusive and collectively exhaustive}}}$$

$$= \varphi_n((N,W)) + \varphi_n((N,R));$$

$$\varphi_n(T) = \varphi_n((T,W)) + \varphi_n((T,R)) \text{ by similar arguments.}$$

Note $\varphi_n(N) + \varphi_n(T) = \sum_{i=1}^{4} \varphi_n(O_i) = 1$.

sum over outcomes of sample space

"Marginal" Frequencies: (Heat Transfer) Knowledge

Interpret Knowledge as a "variable" which can take on two values: W or R.

mutually exclusive
collectively exhaustive

We obtain

$$\varphi_n(W) = \varphi_n((N,W)) + \varphi_n((T,W))$$

and

$$\varphi_n(R) = \varphi_n((N,R)) + \varphi_n((T,R)) .$$

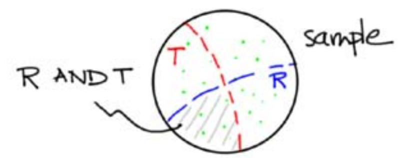
Note $\varphi_n(W) + \varphi_n(R) = 1$.

"Conditional" frequencies: $\mathcal{E}_1, \mathcal{E}_2$

$$\varphi_n(R \mid T) \equiv \frac{\#(R \text{ AND } T)}{\#(T)} \quad \frac{/n}{/n} = \frac{\varphi_n((T,R))}{\varphi_n(T)}$$

frequency of R given T

fraction of {those students who have Taken 2.005} who (also) gave Right answer



Define $\varphi_n(R|T)$, $\varphi_n(T|R)$, $\varphi_n(W|T)$, $\varphi_n(T|W)$, $\varphi_n(R|N)$, $\varphi_n(N|R)$, $\varphi_n(W|N)$, $\varphi_n(N|W)$.

Note

$$\varphi_n(R \mid T) = \frac{\varphi_n((T,R))}{\varphi_n(T)}$$

$$\Downarrow$$

$$\underbrace{\varphi_n((T,R))}_{\text{joint frequency}} = \underbrace{\varphi_n(R \mid T)}_{\text{conditional frequency}} \cdot \underbrace{\varphi_n(T)}_{\text{marginal frequency}}$$

fraction who have Taken 2.005

fraction of {those who have Taken 2.005} who (also) gave Right answer

fraction who have Taken 2.005 AND gave Right answer

Bayes' Theorem: $\mathcal{E}_1, \mathcal{E}_2$

$$\varphi((T,R)) = \varphi(R \mid T) \cdot \varphi(T)$$

$$\varphi((T,R)) = \varphi(T \mid R) \cdot \varphi(R)$$

$$\Downarrow$$

$$\boxed{\varphi(T \mid R) = \frac{\varphi(R \mid T) \cdot \varphi(T)}{\varphi(R)}}$$

correlation vs. causality

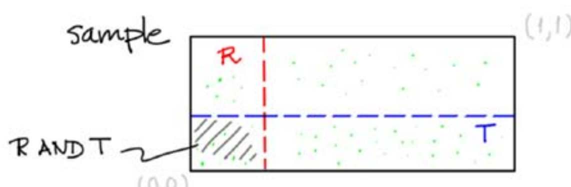
Independence:

events (say) T and R are independent IF

$$\underbrace{\varphi_n(R \text{ AND } T)}_{\varphi((T,R))} = \varphi_n(R) \cdot \varphi_n(T)$$

$$\Leftrightarrow \varphi_n(R \mid T) = \varphi_n(R)$$

sample



from Frequencies to Probabilities

In principle, $\quad n$: sample size

$$\underbrace{P((N,W))}_{\text{probability of }(N,W)} = \lim_{n\to\infty} \varphi_n((N,W))$$

deterministic

"any" subsequence of sample ← random

if limit "exists".

Define for all outcomes of sample space:

$$\varphi_n((N,W)),\ \varphi_n((N,R)),\ \varphi_n((T,W)),\ \varphi_n((T,R))$$
$$\to P((N,W)),\ P((N,R)),\ P((T,W)),\ P((T,R)).$$

Definitions and rules of probability: $\varphi_n \to P$

joint: $P((N,W)), \dots$;

$\mathcal{E}_1$ and $\mathcal{E}_2$: mutually exclusive, collectively exhaustive,

$$P(\mathcal{E}_1 \text{ OR } \mathcal{E}_2) = P(\mathcal{E}_1) + P(\mathcal{E}_2) - P(\mathcal{E}_1 \text{ AND } \mathcal{E}_2);$$

marginal: $P(N), P(T), \dots$;

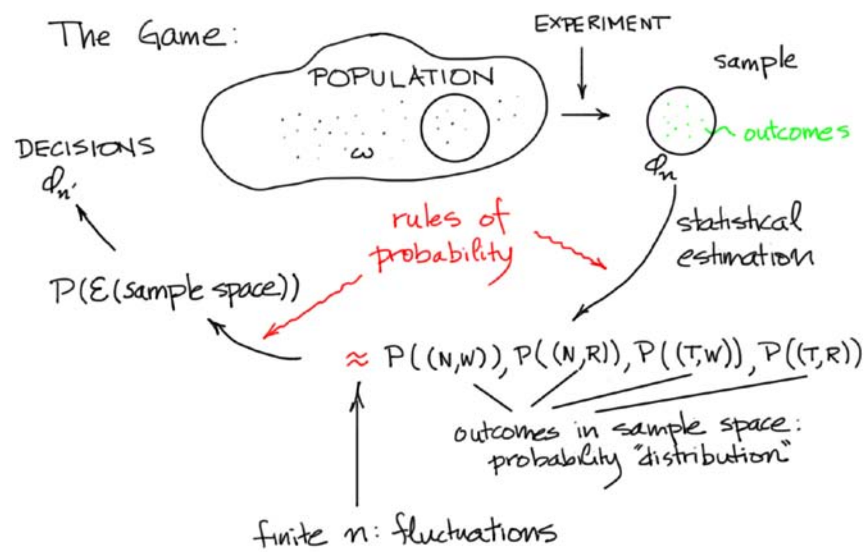
conditional: $P(R|T), P(T|R), P(R|N), \dots$

$$P((T,R)) = P(R|T)\cdot P(T), \dots$$

Bayes Theorem

independence: $\underbrace{P(R \text{ AND } T)}_{P((T,R))} = P(T)\cdot P(R)$

$$\Leftrightarrow \quad P(R|T) = P(R)$$



In practice, must

1) sample population "efficiently" ← Population

2) supplement data with

hypotheses: symmetry, independence, "universality", ... combinatorics, ...

to

synthesize or simplify ← rules of probability

sample spaces or probability distributions.

2.086 Numerical Computation for Mechanical Engineers
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