Homework #3 Solution

1.

(a)

$$\underline{K}\,\underline{U} = \underline{R}$$

$$\underline{K} = \frac{E}{240}\begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}$$

$$\underline{R}_B = \frac{1}{3}\begin{bmatrix} 150 \\ 186 \\ 68 \end{bmatrix} f_2(t) \quad , \quad \underline{R}_S = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} f_1(t)$$

at time = 1,

$$\underline{R} = \begin{bmatrix} 25 \\ 31 \\ 111.33 \end{bmatrix}$$

Solve $\underline{K}\,\underline{U} = \underline{R}$ using $u_1 = 0$

$$\frac{E}{240}\begin{bmatrix} 15.4 & -13 \\ -13 & 13 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 31 \\ 111.33 \end{bmatrix}$$

$$\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1.4233 \\ 1.6288 \end{bmatrix} \times 10^4$$

$$\therefore \quad \underline{u}^{(1)} = \underline{H}^{(1)}\underline{U} = \frac{x}{100} U_2 = 142.33\frac{x}{E}$$

$$\underline{u}^{(2)} = \underline{H}^{(2)}\underline{U} = \frac{1}{E}(14233 + 25.6875x.)$$

$$u(x) = \begin{cases} 142.33\dfrac{x}{E} & \text{over } 0 \le x \le 100 \\ \dfrac{1}{E}(14233 + 25.6875(x-100)) & \text{over } 100 \le x \le 180 \end{cases}$$

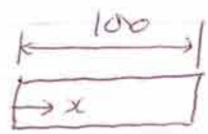
Stresses

$$\tau = E \frac{du}{dx}$$

$$\tau(x) = \begin{cases} 142.33 & \text{over } 0 \le x \le 100 \\ 25.6875 & \text{over } 100 \le x \le 180 \end{cases}$$

To obtain analytical solution, use differential equation and boundary conditions.

element ①

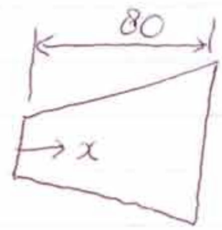


$$EA \frac{d^2 u_1}{dx^2} + A f_x^B = 0$$

$$E \frac{d^2 u_1}{dx^2} + f_x^B = 0$$

$$E \frac{d^2 u_1}{dx^2} + f_2 = E \frac{d^2 u_1}{dx^2} + \frac{1}{2} = 0$$

element ②



$$E \frac{d}{dx}\left(A \frac{du_2}{dx}\right) + A f_x^B = 0$$

$$E \frac{d}{dx}\left(A \frac{du_2}{dx}\right) + \frac{A}{20} = 0$$

Boundary conditions

$$u_1\big|_{x=0} = 0$$

$$EA\frac{du_2}{dx}\bigg|_{x=80} = 100 f_1 = 100$$

$$u_1\big|_{x=100} = u_2\big|_{x=0}$$

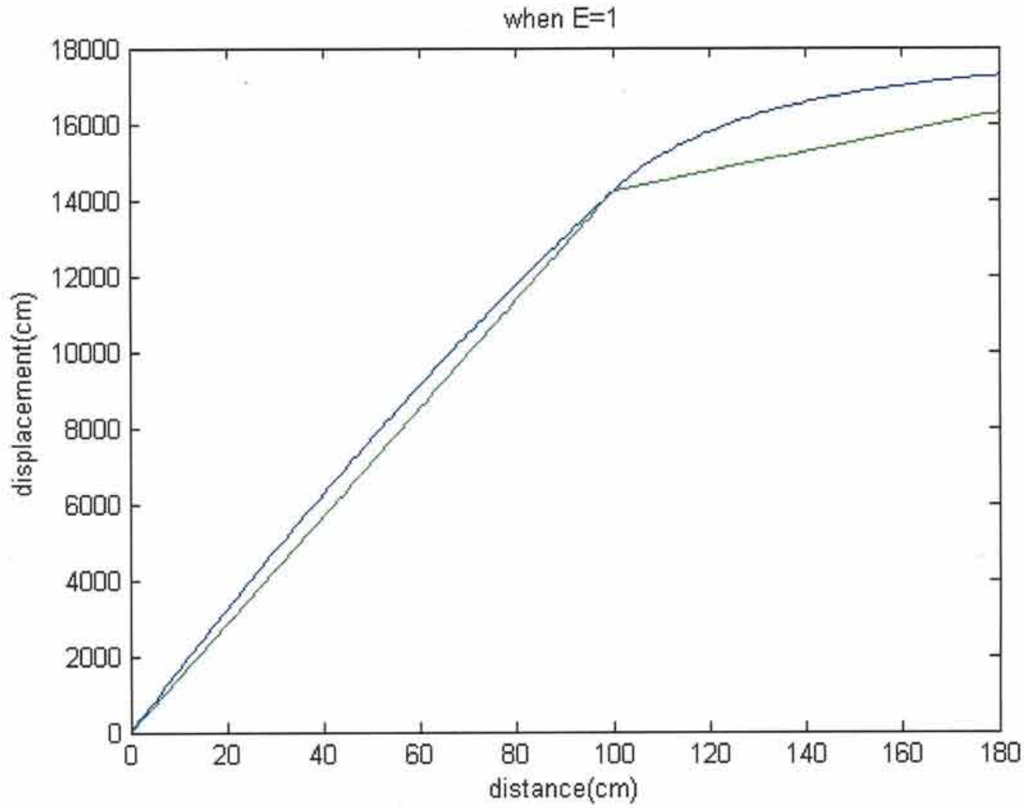
$$\frac{du_1}{dx}\bigg|_{x=100} = \frac{du_2}{dx}\bigg|_{x=0}$$

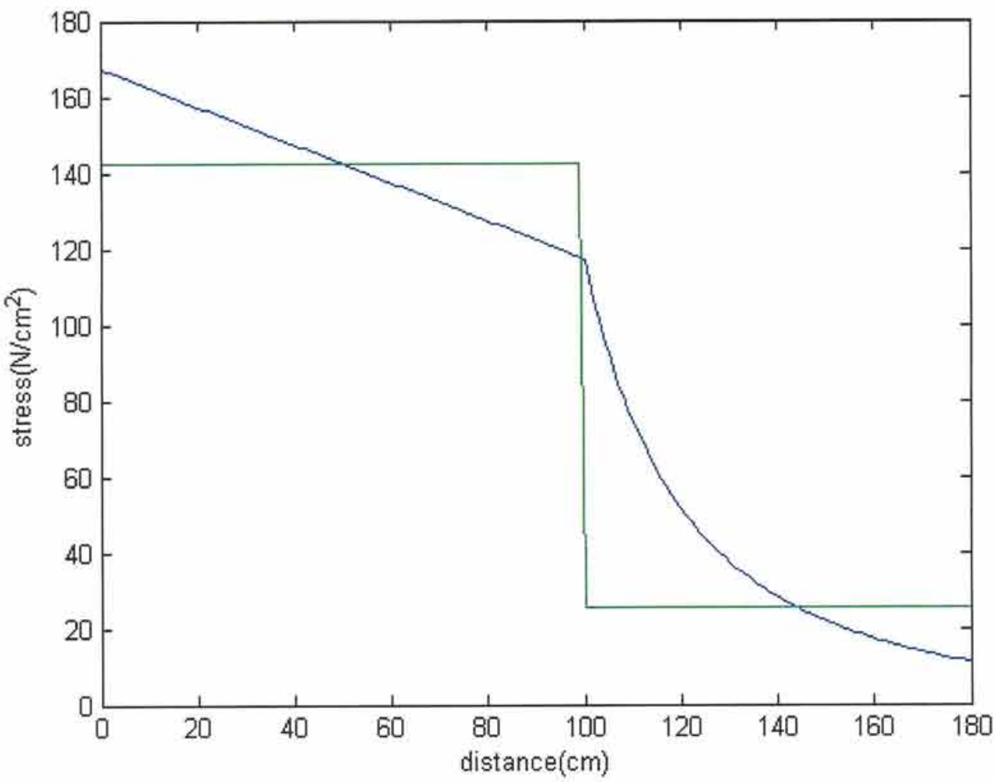
Then we can obtain $u(x)$ and $\tau(x)$

$$u(x) = -\frac{1}{4E}x^2 + \frac{167.33}{E}x \quad \text{over} \quad 0 \le x \le 100$$

$$-\frac{40}{3E}\left(1+\frac{x-100}{40}\right)^2 - \frac{4720}{E}\left(1+\frac{x-100}{40}\right)^{-1} + \frac{18966}{E} \quad \text{over} \quad 100 \le x \le 180$$

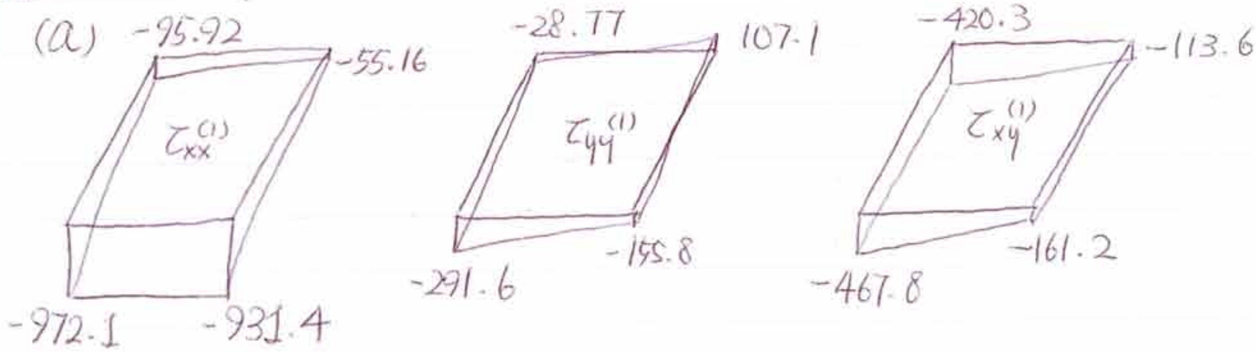
$$\tau(x) = \begin{cases} -\frac{1}{2}x + 167.33 & \text{over} \quad 0 \le x \le 100 \\ -\frac{2}{3}\left(1+\frac{x-100}{40}\right) + 118\left(1+\frac{x-100}{4}\right)^{-2} & \text{over} \quad 100 \le x \le 180 \end{cases}$$


when E=1
18000
16000
14000
12000
10000
8000
6000
4000
2000
0
0
20
40
60
80
100
120
140
160
180
distance(cm)
displacement(cm)


180
160
140
120
100
80
60
40
20
0
0
20
40
60
80
100
120
140
160
180
distance(cm)
stress(N/cm²)

2.

(a)



In order to calculate $\int_{V^{(1)}} \underline{B}^{(1)T} \underline{\tau}^{(1)} dV^{(1)}$

the distribution of $\underline{\tau}^{(1)}$ must be known in advance. In a four node element when displacements are interpolated by nodal values, that is,

$$\underline{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_0 + a_1 x + a_2 y + a_3 xy \\ b_0 + b_1 x + b_2 y + b_3 xy \end{bmatrix}$$

$$\therefore \underline{\tau} = \begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} a_1 + a_3 y \\ b_2 + b_3 x \\ (a_2 + b_1) + a_3 x + b_3 y \end{bmatrix}$$

$$= \frac{E}{1-\nu^2} \begin{bmatrix} (a_1 + \nu b_2) + \nu b_3 x + a_3 y \\ (\nu a_1 + b_2) + b_3 x + \nu a_3 y \\ \frac{1-\nu}{2} \{ (a_2 + b_1) + a_3 x + b_3 y \} \end{bmatrix}$$

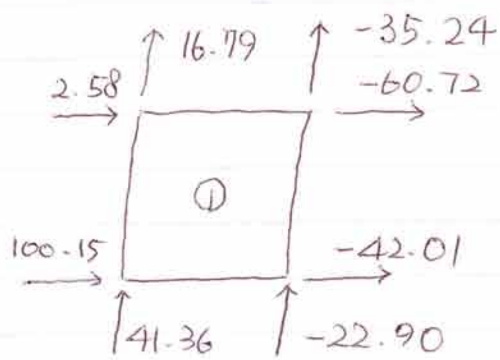
From the given values of $\tau_{xx}^{(1)}$, $\tau_{yy}^{(1)}$ and $\tau_{xy}^{(1)}$ we can obtain by least squares all the constants needed and then

$$\tau_{xx} = -513.65 + 20.37x + 438.11y$$

$$\tau_{yy} = -92.27 + 67.92x + 131.43y$$

$$\tau_{xy} = -290.73 + 153.33x + 23.78y$$

$$\therefore \int_{V^{(1)}} \underline{B}^{(1)T} \underline{\tau}^{(1)} dV^{(1)} = \begin{bmatrix} -60.72 \\ 2.58 \\ 100.15 \\ -42.01 \\ -35.24 \\ 16.79 \\ 41.36 \\ -22.90 \end{bmatrix} = \underline{F}^{(1)} = \begin{bmatrix} -60.72 \\ 2.58 \\ 100.15 \\ -42.01 \\ -35.24 \\ 16.79 \\ 41.36 \\ -22.90 \end{bmatrix}$$



(b) check the balance

$$\Sigma F_x = -60.72 + 2.58 + 100.15 - 42.01 = 0$$

$$\Sigma F_y = -35.24 + 16.79 + 41.36 - 22.90 = 0.01 \simeq 0$$

$$\Sigma M_{(about\ node\ 3)} = -(-60.72)2 + (-35.24)2 - (2.58)2 + (-22.90)2 = 0$$