

Lecture 7 - Finite Element Solution Process, cont'd

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Recall that we have established for the general system some expressions:

$$\mathbf{K}\mathbf{U} = \mathbf{R} \quad (1)$$

$$\mathbf{K} = \sum_m \mathbf{K}^{(m)} \quad ; \quad \mathbf{R} = \mathbf{R}_B + \mathbf{R}_S$$

$$\mathbf{K}^{(m)} = \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)} \quad (2)$$

Then, since $\mathbf{C}^{(m)T} = \mathbf{C}^{(m)}$,

$$\mathbf{K}^{(m)T} = \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)T} \left(\mathbf{B}^{(m)T} \right)^T dV^{(m)} = \mathbf{K}^{(m)}$$

Therefore, \mathbf{K} is symmetric!

$$\mathbf{R}_B^{(m)} = \int_{V^{(m)}} \mathbf{H}^{(m)T} \mathbf{f}^{B(m)} dV^{(m)} \quad (3)$$

For statics, we use $\mathbf{f}^{B(m)}$, but for linear dynamics we must modify it:

$$\mathbf{f}^{B(m)} = \tilde{\mathbf{f}}^{B(m)} - \rho^{(m)} \overbrace{\mathbf{H}^{(m)} \ddot{\mathbf{U}}^{(m)}}^{\ddot{\mathbf{u}}^{(m)}} \quad (4)$$

where $\tilde{\mathbf{f}}^{B(m)}$ is the body force vector excluding inertial loads, and accelerations are interpolated in the same way as displacements.

Now, Eq. (1) becomes

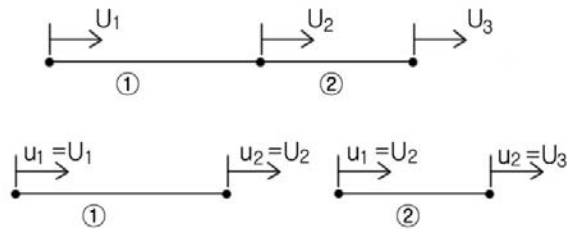
$$\mathbf{M} \ddot{\mathbf{U}} + \mathbf{K} \mathbf{U} = \mathbf{R}$$

$$\mathbf{M} = \sum_m \mathbf{M}^{(m)} \quad ; \quad \mathbf{M}^{(m)} = \int_{V^{(m)}} \mathbf{H}^{(m)T} \rho^{(m)} \mathbf{H}^{(m)} dV^{(m)} \quad (5)$$

We need the initial conditions for $\mathbf{U}(t=0)$ and $\dot{\mathbf{U}}(t=0)$. We will calculate *only the element matrices* corresponding to the element nodes.

Example

Reading assignment: Example 4.5

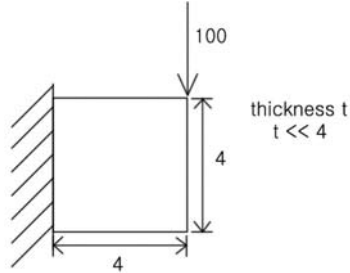


$$\mathbf{K} = \begin{bmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} = \begin{bmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet + \times & \times \\ 0 & \times & \times \end{bmatrix}$$

The nonzero elements in the first term correspond to element 1, and those in the second term correspond to element 2.

Example: Two-Dimensional Analysis

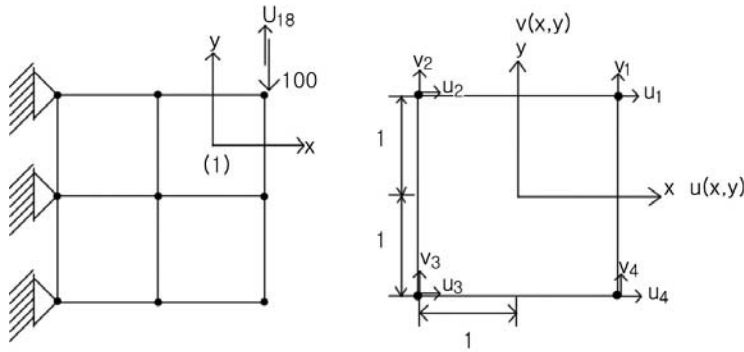
Reading assignment: Example 4.6



This is a thin plate, hence use plane stress assumptions:

$$\mathbf{C} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

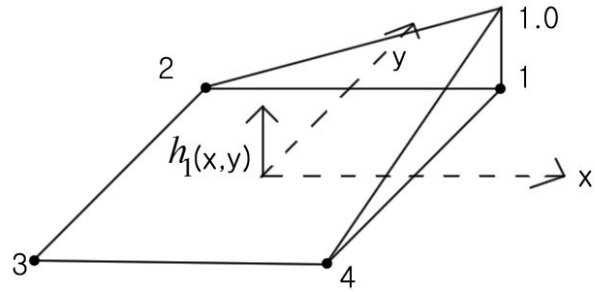
$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{pmatrix}$$



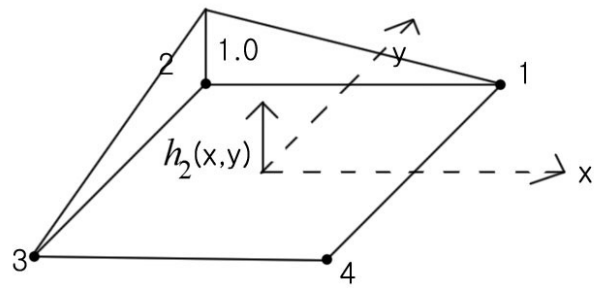
$$\underbrace{\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}}_{2 \times 1} = \underbrace{\mathbf{H}}_{2 \times 8} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{8 \times 1} = \begin{pmatrix} \sum_i^4 h_i u_i \\ \sum_i^4 h_i v_i \end{pmatrix}$$

$$\mathbf{H} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix}$$

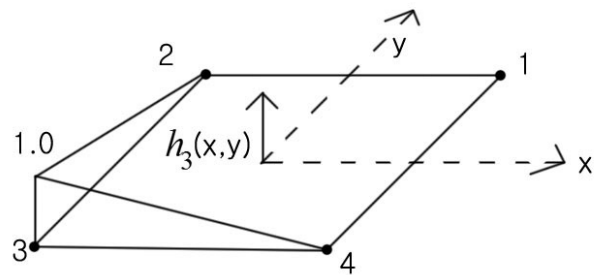
h_i are called "interpolation functions," and are defined as 1 at node i and zero at all other nodes.



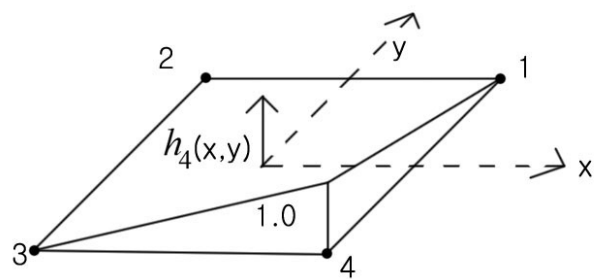
$$h_1(x, y) = \frac{1}{4}(1+x)(1+y)$$



$$h_2(x, y) = \frac{1}{4}(1-x)(1+y)$$



$$h_3(x, y) = \frac{1}{4}(1-x)(1-y)$$



$$h_4(x, y) = \frac{1}{4}(1+x)(1-y)$$

Note that $h_1 + h_2 + h_3 + h_4 = 1$. The interpolation function h_i fulfills the following conditions:

- Constant strain
- Rigid body motion
- Compatibility

$$\mathbf{B} = \begin{bmatrix} h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} \\ h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} & h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} \end{bmatrix}$$

Next, we define $\mathbf{F}^{(m)}$, the element nodal forces:

$$\mathbf{F}^{(m)} = \int_{V^{(m)}} \mathbf{B}^{(m)T} \boldsymbol{\tau}^{(m)} dV^{(m)} \quad (\text{A})$$

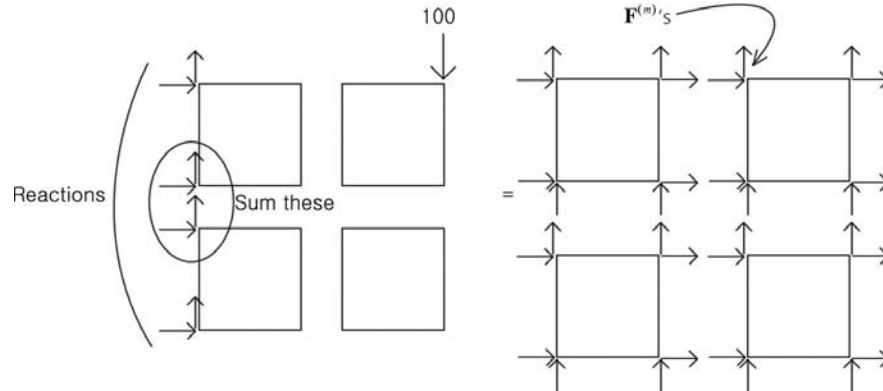
We assume \mathbf{U} has already been calculated, so

$$\boldsymbol{\tau}^{(m)} = \mathbf{C}^{(m)} \mathbf{B}^{(m)} \mathbf{U} \quad (\text{B})$$

Two Important Properties

- I. At any node, the sum of the element nodal point forces is in equilibrium with the externally applied nodal loads; that is:

$$\sum_m \mathbf{F}^{(m)} = \mathbf{R} \quad ; \quad \sum_m \mathbf{F}^{(m)} = \mathbf{F}$$



Why do we have this property? Use (A) and (B):

$$\left[\sum_m \int_{V^{(m)}} \mathbf{B}^{(m)T} \mathbf{C}^{(m)} \mathbf{B}^{(m)} dV^{(m)} \right] \mathbf{U} = \mathbf{K} \mathbf{U} = \mathbf{R}$$

- II. Each element (m) is in equilibrium under its nodal forces.

Proof:

$$\overline{\mathbf{U}}_r^T \mathbf{F}^{(m)} = \int_{V^{(m)}} \overbrace{\overline{\mathbf{U}}_r^T \mathbf{B}^{(m)T}}^{\overline{\boldsymbol{\varepsilon}}_r^{(m)T}=0} \boldsymbol{\tau}^{(m)} dV^{(m)}$$

In this expression, $\overline{\mathbf{U}}_r^T$ represents a rigid body motion. Now we use rigid body motions corresponding to translations and rotations.

In general, to obtain the exact analytical solution, the following should be satisfied:

- Differential equilibrium both in the volume, and on the surface
- Compatibility
- Stress-strain law

For a properly formulated finite element procedure, the displacement interpolation functions ensure that compatibility is satisfied. The stress-strain law is also enforced; however, differential equilibrium is not satisfied at every point in the continuum. Rather, equilibrium is only satisfied at every node and on each element (see properties I and II above). As such, the finite element solution is approximate, and this approximation becomes more accurate as the mesh is refined, or the interpolation order of the element is increased.

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