2.092/2.093 — Finite Element Analysis of Solids & Fluids I Fall '09

Lecture 10 - Nonlinear Finite Element Analysis of Solids & Structures

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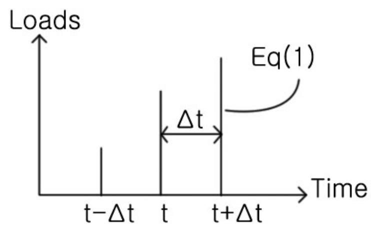
Reading assignment: Sections 6.1, 8.4.1

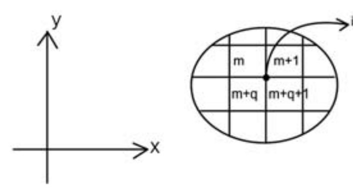
Discretization of the variational formulation leads to the following equilibrium statement:

$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R} \tag{1}$$

$^{t+\Delta t}\boldsymbol{F}$ = nodal forces corresponding to element stresses at time $t+\Delta t$
$^{t+\Delta t}\boldsymbol{R}$ = external loads applied at the nodes at time $t+\Delta t$

It is assumed that the solution is known up to time t. The problem is to find $^{t+\Delta t}\boldsymbol{F}$, given $^{t+\Delta t}\boldsymbol{R}$ where:



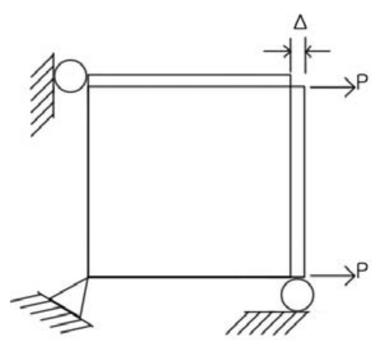


$$ {}^{t+\Delta t}\boldsymbol{F}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)T} \left({}^{t+\Delta t}\boldsymbol{\tau}^{(m)} \right) dV^{(m)} $$

In linear analysis, $V^{(m)}$, $B^{(m)}$ are constant. In general nonlinear analysis, $V^{(m)}$, $B^{(m)}$ are functions of time.

Types of Analysis

I. Linear analysis (e.g. response of an airplane, car under operating loads)



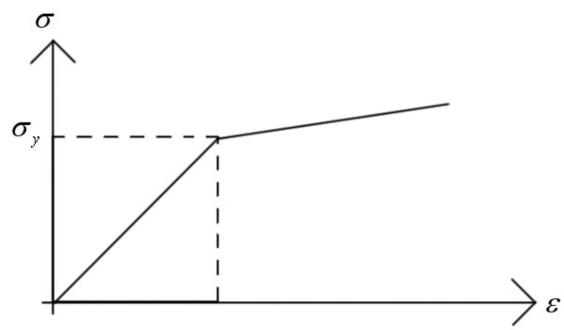
(a) $\frac{\Delta}{L} =$ strain < 0.04
(b) $\tau = E\varepsilon$, E is a constant
(c) $\Delta \rightarrow$ also small

$$ {}^{t+\Delta t}\boldsymbol{F}^{(m)} = \boldsymbol{K}^{(m)}\, {}^{t+\Delta t}\boldsymbol{U}^{(m)} \rightarrow \boldsymbol{K}\, {}^{t+\Delta t}\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} $$

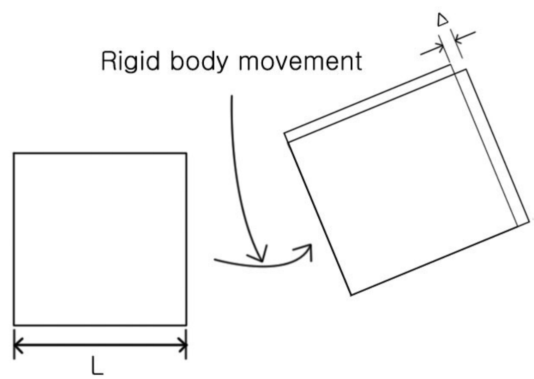
$$ \boldsymbol{K} = \sum_m \boldsymbol{K}^{(m)} \quad \text{(Constant)} $$

II. Materially-nonlinear-only (often found in geomechanics, e.g. sand, rocks, tunnel building)

(a) and (c) hold here as well, but the stress-strain relation is nonlinear:



III. Large displacements & small strains (e.g. buckling analysis of shell structures)

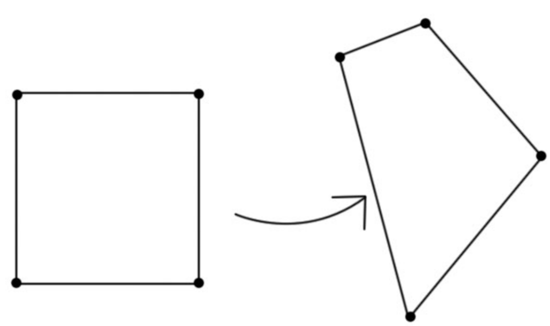


(a) $\frac{\Delta}{L} \leq 0.04$ (strains are still small)

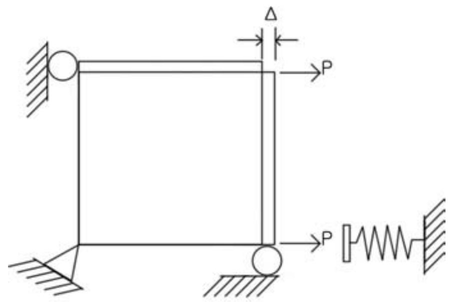
(b) Displacements are large

(c) Stress-strain relation may be linear or nonlinear

IV. Large displacements & large strains (e.g. rubber O-rings, metal forming, crash analysis)



V. Contact: Change in boundary conditions



These are difficult to solve. The boundary conditions change when P is large to make the element take contact with the spring. The cases II to V may contain nonlinearities (a system could have combinations). Dynamic analysis can also be included in the system analysis.

To solve ${}^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R}$ in general nonlinear analysis, assume that we have already solved ${}^{t}\boldsymbol{F} = {}^{t}\boldsymbol{R}$, and we also know ${}^{t}\boldsymbol{U}$, ${}^{t}\boldsymbol{\tau}$. Then we can write

$$ {}^{t+\Delta t}\boldsymbol{F} = {}^{t}\boldsymbol{F} + \overset{?}{\boldsymbol{F}} = {}^{t+\Delta t}\boldsymbol{R} $$

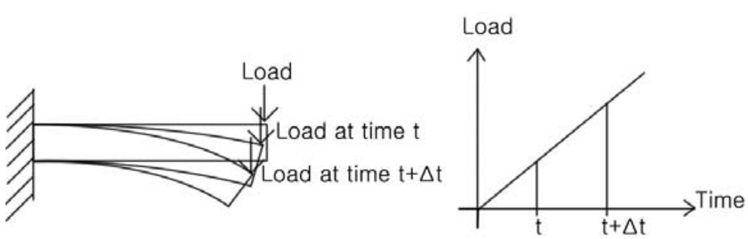
where ${}^{t+\Delta t}\boldsymbol{R}$ is known *a priori* and $\overset{?}{\boldsymbol{F}}$ is what we are seeking. Then,

$$ \overset{?}{\boldsymbol{F}} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F} $$

$$ \overset{?}{\boldsymbol{F}} \doteq {}^{t}\boldsymbol{K}\Delta\boldsymbol{U} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F} \tag{2} $$

where ${}^{t}\boldsymbol{K}$ is the tangent stiffness matrix at time t. This gives us the increment in displacements.

Example



We now solve Eq. (2) for $\Delta\boldsymbol{U}$. Then,

$$ {}^{t+\Delta t}\boldsymbol{U} \doteq {}^{t}\boldsymbol{U} + \Delta\boldsymbol{U} $$

Iteration is needed; the Newton-Raphson technique is widely used. Iterate for $i = 1, 2, \ldots$ until convergence is reached.

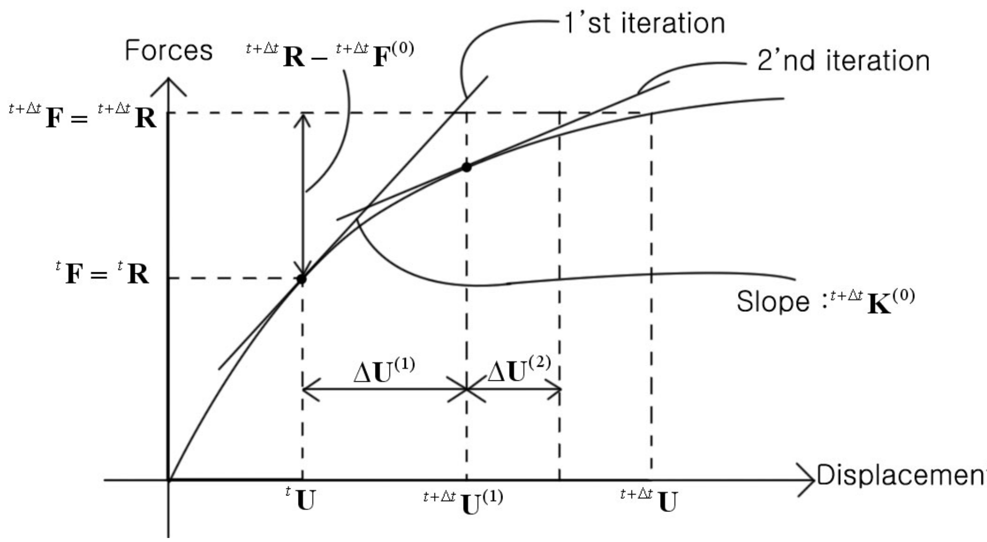
$$ {}^{t+\Delta t}\boldsymbol{K}^{(i-1)}\Delta\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(i-1)} \tag{A} $$

Using

$$ {}^{t+\Delta t}\boldsymbol{U}^{(i)} = {}^{t+\Delta t}\boldsymbol{U}^{(i-1)} + \Delta\boldsymbol{U}^{(i)} \tag{B} $$

and the initial conditions

$$ {}^{t+\Delta t}\boldsymbol{K}^{(0)} = {}^{t}\boldsymbol{K} \quad ; \quad {}^{t+\Delta t}\boldsymbol{F}^{(0)} = {}^{t}\boldsymbol{F} \quad ; \quad {}^{t+\Delta t}\boldsymbol{U}^{(0)} = {}^{t}\boldsymbol{U} \tag{C} $$



For $i = 1$, Eq. (A) is Eq. (2) with $\Delta \boldsymbol{U}^{(1)} = \boldsymbol{U}$. Find $^{t+\Delta t}\boldsymbol{U}^{(1)}$, then find the new element forces $^{t+\Delta t}\boldsymbol{F}^{(1)}$.

$$\text{1st iteration: } {}^{t}\boldsymbol{K}\Delta\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F}$$

$$\text{2nd iteration: } {}^{t}\boldsymbol{U} + \Delta\boldsymbol{U}^{(1)} = {}^{t+\Delta t}\boldsymbol{U}^{(1)} \qquad \text{(See Eq. B)}$$

$$^{t+\Delta t}\boldsymbol{K}^{(1)}\Delta\boldsymbol{U}^{(2)} = {}^{t+\Delta t}\boldsymbol{R} - {}^{t+\Delta t}\boldsymbol{F}^{(1)}$$

$^{t+\Delta t}\boldsymbol{F}^{(1)}$ is calculated using $^{t+\Delta t}\boldsymbol{U}^{(1)}$ and the material law.

If increments in displacements become very small ($\sim 10^{-6}$, 10^{-8}), or $^{t+\Delta t}\boldsymbol{R} - {}^{t}\boldsymbol{F}^{(i-1)}$ gets very small, we stop iterating. When these conditions occur, we know that we have satisfied $^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R}$. However, the system will not converge if the time steps are too large.

By these procedures, we have satisfied the following conditions:

- Compatibility
- Stress-strain laws
- Equilibrium is satisfied only for each finite element and each node

The accuracy of the analysis depends on

- Fineness of the mesh, elements used
- Solution of $^{t+\Delta t}\boldsymbol{F} = {}^{t+\Delta t}\boldsymbol{R}$

Historically, it was very expensive to update the $\boldsymbol{K}$ matrix every iteration, to keep using $^{t+\Delta t}\boldsymbol{K}^{(i-1)}$. So, the $\boldsymbol{K}$ matrix was set up once in the beginning and kept constant during the iteration. Using this method, more iterations are needed but we perform fewer calculations per iteration. The $\boldsymbol{K}$ matrix can be "somewhat wrong", but we must calculate $^{t+\Delta t}\boldsymbol{F}^{(i-1)}$ correctly in each iteration. This procedure is known as the modified Newton-Raphson method.

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